Function Analysis

Goal

There are two goals of this section:

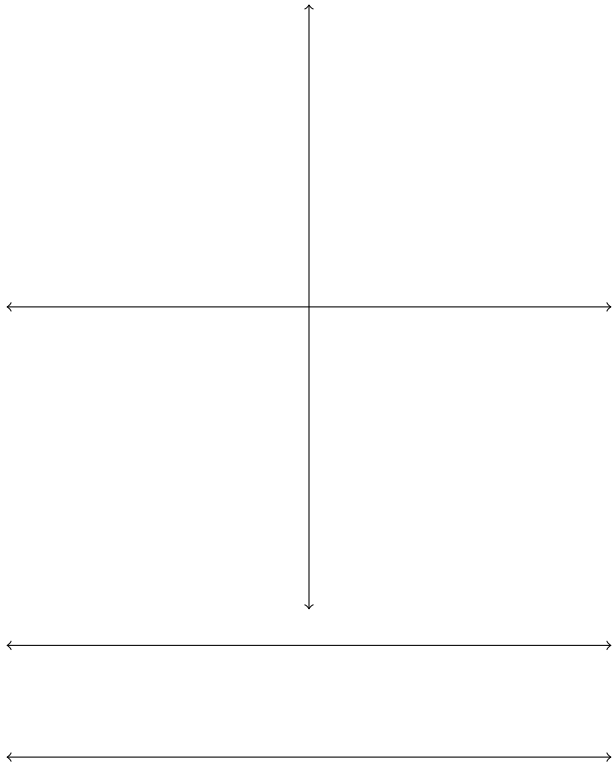
1. It will be important for us to have intuition about the overall shape of a graph and how that shape is related to the first and second derivatives.
2. Spending time sketching graphs is an excellent way to synthesize all of the relationships between a functions, its derivatives, infinite limits, and its overall shape.

Introduction

In general, we will be given a function and asked to sketch its graph. To do this, we will have to identify some key features of the graph (see guidelines below). Let's first try to sketch the graph of a function where some useful information is given.

Example 1. Sketch the graph of the function that has the following properties.

1. $f(-5) = 0, f(-3) = -3, f(-2) = 0$
2. $f(-1.5) = .5, f(-.5) = 1, f(1.5) = 2.5$
3. $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 3} f(x) = \infty$
4. $\lim_{x\to\infty} f(x) = 1$ and $\lim_{x\to-\infty} f(x) = 1$
5. $f'(-3)$ undefined
6. $f'(1.5) = 0, f'(-1.5) = 0$
7. $f'(x) > 0$ on $(-3, -1.5)$, $(-1.5, 0)$, $(1.5, 3)$
8. $f'(x) < 0$ on $(-\infty, -3), (0, 1.5)$, $(3, \infty)$
9. $f''(x) > 0$ on $(-1.5, 0), (0, 3)$, $(3, \infty)$
10. $f''(x) < 0$ on $(-\infty, -3)$, $(-3, -1.5)$



Guidelines for Sketching Graphs of Functions

The following checklist is intended as a guide to sketching a curve $y = f(x)$ by hand. Not every item is relevant to every function.

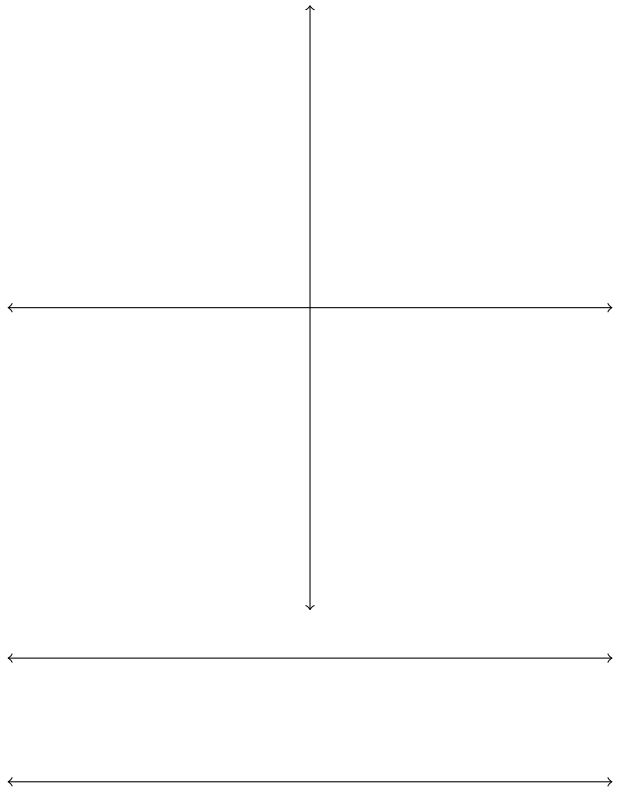
1. Find x-intercepts. (Finding x-intercepts is not always easy and an attempt to find them should be abandoned if too difficult.)
2. Find y-intercept.
3. Identify vertical asymptotes.
4. Determine end behavior by computing limits of $f(x)$ as $x \to \infty$ and $x \to -\infty$ (Does graph have any horizontal asymptotes?).
5. Find critical numbers, determine intervals of increase and decrease, and identify any relative extrema. Plot the points corresponding to the critical numbers (to find y-values, plug the corresponding x-value into the original function).
6. Find x-values where $f''(x) = 0$ or is undefined, determine intervals of concavity, and identify any inflection points. Again, plot the corresponding points (to find y-values, plug the corresponding x-value into the original function).

(Use this space below if you run out of room on the following pages.)

 Written by D.C. Ernst

Example 2. Sketch the graph of the following functions.

(a) $f(x) = \dfrac{2(x^2 - 9)}{x^2 - 4}$



(b) $g(x) = \dfrac{-x}{(x^2-1)^2}$