Integration by Parts

Goal

In this section, we will introduce the method of integration by parts for evaluating definite and indefinite integrals.

Motivation and Background

Recall that integrating products, quotients, and compositions is "hard." The method of substitution takes a long way to integrating functions consisting of products, quotients, and compositions of simpler functions, but there are still lots of functions we cannot handle. So, we need more techniques! The method of *integration by parts* is useful for integrating things of the form:

$$\int f'(x)g(x)\ dx$$

Every time we have a formula for the derivative of a function, we have a corresponding integration formula. Furthermore, you may not have noticed, but every differentiation rule has a corresponding integration rule. For example, loosely speaking, the chain rule for derivatives corresponds to u-substitution for integrals. Does the integrand above remind you of any of our differentiation rules?

Integration by Parts

Theorem 1 (Integration by Parts)**.** Suppose f and g are differentiable functions. Then

$$\boxed{\int f'(x)g(x)\ dx = f(x)g(x) - \int f(x)g'(x)\ dx}$$

Proof. By the Product Rule

$$\frac{d}{dx}[f(x)g(x)] = \underline{\hspace{6cm}}.$$

Solving for $f'(x)g(x)$, we obtain

$$f'(x)g(x) = \underline{\hspace{6cm}}.$$

Then if we take the indefinite integral of both sides of the above equation, we have

$$\underline{\hspace{6cm}} = \underline{\hspace{9cm}}.$$

But

$$\int \frac{d}{dx}[f(x)g(x)]\ dx = \underline{\hspace{6cm}},$$

which implies that

$$\int f'(x)g(x)\ dx = \underline{\hspace{9cm}},$$

as desired. $\square$

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Note 2. Most people write the formula for integration by parts as

$$\boxed{\int u\ dv = uv - \int v\ du,}$$

where $u = g(x)$ and $dv = f'(x)dx$.

Important Note 3. To use integration by parts, we need to identify

(i) u;

(ii) dv (it must be something we can integrate).

Then we must find

(iii) du (by differentiation);

(iv) v (by integration).

Note that the formula for integration by parts is what one would expect if we are dealing with a definite integral:

$$\boxed{\int_a^b u\ dv = \underline{\hspace{6cm}}.}$$

Examples

Let's do some examples.

Example 4. Integrate each of the following.

(a) $\displaystyle\int xe^{-x}\ dx$

(b) $\displaystyle\int x^2 \sin(x)\ dx$

(c) $\displaystyle\int_0^1 \arctan(x)\ dx$

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Comments

As time goes on, our proficiency at picking the correct u and dv will increase. Here is a list of "suggestions" for common integrals using integration by parts.

1. For
$$\int x^n e^{ax}\, dx, \quad \int x^n \sin(ax)\, dx, \quad \int x^n \cos(ax)\, dx$$
let $u = x^n$ and $dv = e^{ax}\, dx$, $\sin(ax)\, dx$, or $\cos(ax)\, dx$.
2. For
$$\int x^n \ln(x)\, dx, \quad \int x^n \arcsin(ax)\, dx, \quad \int x^n \arctan(ax)\, dx$$
let $u = \ln(x)$, $\arcsin(ax)$, or $\arctan(x)$ and $dv = x^n\, dx$.
3. For
$$\int e^{ax} \sin(bx)\, dx, \quad \int e^{ax} \cos(bx)\, dx$$
either choice will work and regardless of your choice, you will have to do a "feedback loop" (see next example).

Note 5. You can use the acronym LIATE to help you choose what to let u equal when doing integration by parts.

Logarithmic functions

Inverse trigonometric functions

Algebraic functions

Trigonometric functions

Exponential functions

Another Example

Here is one more example, which exhibits what I call a "feedback loop".

Example 6. Integrate $\int e^x \cos(x)\, dx$.