Homework 4

Discrete Mathematics

Please review the *Rules of the Game* from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition.You are highly encouraged to seek out assistance by asking questions in our Q&A Discussion Board in Canvas. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to three late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please $\boxed{\text{box}}$ your final answer.

1. Imagine we have $n \geq 1$ people competing in a contest where ties are allowed.
   (a) Explain why the number of final rankings is given by
   $$\sum_{k=1}^{n} k! \left\{ {n \atop k} \right\} := 1! \left\{ {n \atop 1} \right\} + 2! \left\{ {n \atop 2} \right\} + \cdots + n! \left\{ {n \atop n} \right\},$$
   where $\left\{ {n \atop k} \right\}$ is a Stirling number (see Homework 2).
   (b) Now, imagine that the people are numbered 1 through n. How many final rankings are possible under the proviso that person i's ranking is less than or equal to person $(i+1)$'s ranking? Such rankings are called *weakly increasing*.
2. OMITTED
3. More coffee! Unfortunately, my favorite coffee shop from Problem 1.14 and Homework 1.2 is closed today. However, there is another coffee shop a little further away from my starting position. This second coffee shop is 5 blocks North and 5 blocks East from my starting position. Let's call this second coffee shop, B.
   (a) Assuming I only walk East or North, how many different routes can I take to get to coffee shop B from my original starting location?

(b) Doh! That pesky construction site still exists at 3 blocks East and 2 blocks North from my start location. How many different routes can I take to get to coffee shop B from my original starting location if I avoid this intersection?

4. A math major has six more MAT/STA courses to take in her senior year, and wishes to take no more than four in a semester. Assume she has all necessary prerequisites for her remaining classes.

   (a) In how many ways can she do this if the courses to be taken are MAT 226, 238, 239, 316, 365 and STA 275, which are all offered in both semesters?

   (b) In how many ways can this be done if instead she needs MAT 226, 239, 316 and the Spring-only courses MAT 441, STA 371 and STA 474?

5. Recall the definition of a composition of n from Homework 2. Prove that the number of compositions of n with k parts is $\binom{n-1}{k-1}$.

6. Define a **multiset** M to be an unordered collection of elements that may be repeated. To distinguish between sets and multisets, we will use the notation $\{\}$ versus $\{\!\{\}\!\}$. For example, $M = \{\!\{a,a,a,b,c,c\}\!\}$ is a multiset (not to be confused with the set containing the set $\{a,b,c\}$; mathematical notation is both awesome and frustrating at times). The primes occurring in the prime factorization of a natural number greater than 1 is an example of multiset. The cardinality of a multiset is its number of elements counted with multiplicity. So, in our example, $|M| = 3+1+2 = 6$. If S is a set, then M is a **multiset on** S if every element of M is an element of S. For $n \geq 1$ and $k \geq 0$, define

$$\left(\!\!\binom{n}{k}\!\!\right) := \text{number of multisets on } [n] \text{ of size } k.$$

   Prove that

$$\left(\!\!\binom{n}{k}\!\!\right) = \binom{n+k-1}{k}.$$

   Hint: Here is one possible approach. Construct a bijection from the collection of multisets on $[n]$ of size k to the collection of subsets of $[n+k-1]$ of size k. To do this, assume your multisets on $[n]$ are canonically written in nondecreasing order, say $\{\!\{m_1, m_2, \ldots, m_k\}\!\}$, where $m_1 \leq m_2 \leq \cdots \leq m_k$. Try to map each such multiset to a k-subset of $[n+k-1]$. You will need to make sure the elements in your image set are distinct.