Problem 96. In a certain kind of tournament, every player plays every other player exactly once and either wins or loses (there are no ties). Define a *top player* to be a player who, for every other player x, either beats x or beats a player y who beats x. (There may be more than one top player.) Prove that every n-player tournament has a top player. *Hint:* Use induction. For the inductive step, start with a tournament with $k + 1$ players and remove a single player that has the lowest number of wins. There might be lots of players tied for lowest number of wins, in which case just pick one of them at random to remove.

We will prove this by induction on the # of players in tournament. It will be helpful to have some notation. If x beats y, we can represent this via:

$$x \longrightarrow y$$

Base case: In a tournament w/ 2 players, there is one game, say:

$$x \longrightarrow y$$

In this case, x is top player.

Inductive Step: Assume that every tournament w/ k players has at least one top player.

Now, consider a tournament w/ $k+1$ players.

After all games are played, each player has some # of wins and some # of losses. Choose any player, say p, that has the most losses (if there is a tie, choose at random).

Temporarily ignore p and all the games that p has played. Now, we have a tournament w/ k players. By our inductive hypothesis, this smaller tournament has a top player, say t.

Now, consider the original tournament w/ all $k+1$ players. Here are the possibilities:

Case 1: t beats p:

$$t \longrightarrow p$$

In this case, t is still a top player.

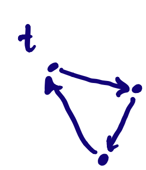
Case 2: p beats t:

$$t \longleftarrow p$$

At this point, there are 2 sub-possibilities:

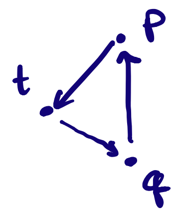
(a) p beats t and p beat all players that t beat:

There might be players that t lost to missing in pic, but that's ok. For any such player, we have:



However, notice that p won more games than t in this scenario. This contradicts our assumption about p. So, this can't happen!

(b) p beats t and there exists a player q s.t. t beat q and q beat p:



looks like "Rock, Paper, Scissors"

In this case, t is still a top player.

The upshot is that t is top player in both tournaments. But notice this specific top player depended on the removal of a "bad" player. It's not true in general that a top player in smaller tournament will remain top in larger tournament.