I counted everything. I counted the steps to the road, the steps up to church, the number of dishes and silverware I washed ... anything that could be counted, I did.

Katherine Johnson, mathematician

Chapter 9

Cardinality

In this chapter, we will explore the notion of cardinality, which formalizes what it means for two sets to be the same "size".

9.1 Introduction to Cardinality

What does it mean for two sets to have the same "size"? If the sets are finite, this is easy: just count how many elements are in each set. Another approach would be to pair up the elements in each set and see if there are any left over. In other words, check to see if there is a one-to-one correspondence (i.e., bijection) between the two sets.

But what if the sets are infinite? For example, consider the set of natural numbers $\mathbb{N}$ and the set of even natural numbers $2\mathbb{N} := \{2n \mid n \in \mathbb{N}\}$. Clearly, $2\mathbb{N}$ is a proper subset of $\mathbb{N}$. Moreover, both sets are infinite. In this case, you might be thinking that $\mathbb{N}$ is "larger than" $2\mathbb{N}$. However, it turns out that there is a one-to-one correspondence between these two sets. In particular, consider the function $f : \mathbb{N} \to 2\mathbb{N}$ defined via $f(n) = 2n$. It is easily verified that f is both injective and surjective. In this case, mathematics has determined that the right viewpoint is that $\mathbb{N}$ and $2\mathbb{N}$ do have the same "size". However, it is clear that "size" is a bit too imprecise when it comes to infinite sets. We need something more rigorous.

Definition 9.1. Let A and B be sets. We say that A and B have the same **cardinality** if there exists a bijection between A and B. In this case, we write $\boxed{\operatorname{card}(A) = \operatorname{card}(B)}$.

Note that we have not defined $\operatorname{card}(A)$ by itself. Doing so would not be too difficult for finite sets, but making such a notation precise in general is tricky business. When we write $\operatorname{card}(A) = \operatorname{card}(B)$ (and later $\operatorname{card}(A) \leq \operatorname{card}(B)$ and $\operatorname{card}(A) < \operatorname{card}(B)$), we are asserting the existence of a certain type of function from A to B.

If f is a bijection from A to B, then by Theorem 8.77, f^{-1} is a bijection from B to A. Either one of these functions can be utilized to prove that $\operatorname{card}(A) = \operatorname{card}(B)$. This idea is worth keeping in mind as you tackle problems in this chapter. In particular, you might have an easier time creating a bijection between two sets in one direction over the other. This is often a limitation of the human mind as to opposed to some fundamental mathematical difficulty.

Example 9.2. Let $A = \{1,2,3,4,5\}$ and $B = \{\text{apple}, \text{banana}, \text{cherry}, \text{dragon fruit}, \text{elderberry}\}$. The function $f : A \to B$ given by

$$f = \{(1, \text{apple}), (2, \text{banana}), (3, \text{cherry}), (4, \text{dragon fruit}), (5, \text{elderberry})\}$$

is a bijection from A to B, and hence $\operatorname{card}(A) = \operatorname{card}(B)$. Note that this is not the only bijection from A to B. In fact, there are $5! = 120$ bijections from A to B, one of which is the function f we defined above. The inverse of each bijection from A to B is a bijection from B to A. We could also use any of of these bijections to verify that $\operatorname{card}(A) = \operatorname{card}(B)$.

Example 9.3. Define $f : \mathbb{Z} \to 6\mathbb{Z}$ via $f(n) = 6n$. It is easily verified that f is both injective and surjective, and hence $\operatorname{card}(\mathbb{Z}) = \operatorname{card}(6\mathbb{Z})$. We could also utilize the inverse function $f^{-1} : 6\mathbb{Z} \to \mathbb{Z}$ given by $f^{-1}(n) = \frac{1}{6}n$ to show that $\mathbb{Z}$ and $6\mathbb{Z}$ have the same cardinality.

Example 9.4. Let $\mathbb{R}^+$ denote the set of positive real numbers and define $f : \mathbb{R} \to \mathbb{R}^+$ via $f(x) = e^x$. As you are likely familiar with, this exponential function is a bijection, and so $\operatorname{card}(\mathbb{R}) = \operatorname{card}(\mathbb{R}^+)$. Similar to the previous example, we could also use the inverse function $f^{-1} : \mathbb{R}^+ \to \mathbb{R}$ given by $f^{-1}(x) = \ln(x)$ to show that these two sets have the same cardinality.

The previous two examples illustrate an important distinction between finite sets and infinite sets, namely infinite sets can be in bijection with proper subsets of themselves! Theorems 9.23 and 9.31 will make this idea explicit.

Example 9.5. Let $m, n \in \mathbb{N} \cup \{0\}$. A North-East lattice path from $(0,0)$ to (m,n) is path in the plane from $(0,0)$ to (m,n) consisting only steps either one unit North or one unit East. Note that every lattice path from $(0,0)$ to (m,n) consists of a total of $m+n$ steps. Figure 9.1 shows a North-East lattice path from $(0,0)$ to $(4,3)$. Let $\mathcal{L}_{m,n}$ denote the set of North-East paths from $(0,0)$ to (m,n). For example, the North-East lattice path given in Figure 9.1 is an element of $\mathcal{L}_{4,3}$. A binary string of length k is an ordered list of consisting of k entries where each entry is either 0 or 1. For example, 0101100 and 0101001 are two different binary strings of length 7. Let $\mathcal{S}_k$ denote the set of binary strings of length k. For example, $\mathcal{S}_3 = \{000, 100, 010, 001, 110, 101, 011, 111\}$. We claim that there is a bijection between $\mathcal{L}_{m,n}$ and $\mathcal{S}_{m+n}$. One such bijection is given by mapping a lattice path to the string that results by assigning each East step to 0 and each North step to 1 as we travel the path from $(0,0)$ to (m,n). Using this construction, the lattice path in Figure 9.1 would get mapped to the binary string 0101100. Since no two lattice paths will map to the same string, our mapping is injective. Given a string in $\mathcal{S}_{m+n}$, it is easy to find the lattice path in $\mathcal{L}_{m,n}$ that maps to it, and so our function is also surjective. Thus, our mapping is a bijection between $\mathcal{L}_{m,n}$ and $\mathcal{S}_{m+n}$. We have shown that $\operatorname{card}(\mathcal{L}_{m,n}) = \operatorname{card}(\mathcal{S}_{m+n})$.

When approaching Part (d) of the next problem, try creating a linear function $f : (a,b) \to (c,d)$. Drawing a picture should help.

Problem 9.6. Prove each of the following. In each case, you should create a bijection between the two sets. Briefly justify that your functions are in fact bijections.

(a) $\operatorname{card}(\{a,b,c\}) = \operatorname{card}(\{x,y,z\})$

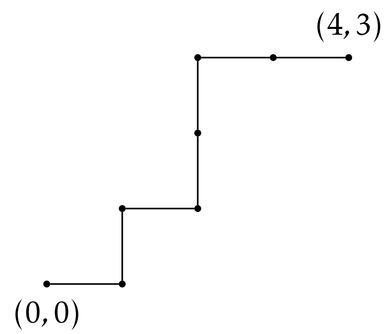

Figure 9.1: A North-East lattice path from $(0,0)$ to $(4,3)$.

(b) $\text{card}(\mathbb{N}) = \text{card}(\{2n+1 \mid n \in \mathbb{N}\})$

(c) $\text{card}(\mathbb{N}) = \text{card}(\mathbb{Z})$

(d) $\text{card}((a,b)) = \text{card}((c,d))$ (where (a,b) and (c,d) are intervals)

(e) $\text{card}(\mathbb{N}) = \text{card}(\{\frac{1}{2^n} \mid n \in \mathbb{N}\})$

Problem 9.7. If A is a set, do A and $A \times \{x\}$ have the same cardinality? Justify your answer.

Problem 9.8. Let $\mathcal{D}_n$ denote the collection of North-East lattice paths from $(0,0)$ to (n,n) that never drop below the line $y = x$. These types of lattice paths are often called Dyck paths after the German mathematician Walther Franz Anton von Dyck (1856–1934). A sequence of parentheses is balanced if it can be parsed syntactically. In other words, there should be the same number of open parentheses "(" and closed parentheses ")", and when reading from left to right there should never be more closed parentheses than open. For example, ()()() and ()(()) are balanced parenthesizations consisting of three pairs of parentheses while ())(() and ()(()(are not balanced. Let $\mathcal{B}_n$ denote the collection of balanced parenthesizations consisting of n pairs of parentheses. For example, $\mathcal{B}_3 = \{()()(), ()(()), (()()), (())(), ((()))\}$.

(a) Find all Dyck paths in $\mathcal{D}_3$.

(b) Prove that $\text{card}(\mathcal{D}_n) = \text{card}(\mathcal{B}_n)$.

For Part (b) of the next problem, start by defining $\varphi : \mathcal{F} \to \mathcal{P}(\mathbb{N})$ so that $\varphi(f)$ yields a subset of $\mathbb{N}$ determined by when f outputs a 1.

Problem 9.9. Let $\mathcal{F}$ denote the set of functions from $\mathbb{N}$ to $\{0,1\}$.

(a) Describe at least three functions in $\mathcal{F}$.

(b) Prove that $\mathcal{F}$ and $\mathcal{P}(\mathbb{N})$ have the same cardinality.

Our first theorem concerning cardinality will likely not come as a surprise.

Theorem 9.10. Let A, B, and C be sets.

(a) $\text{card}(A) = \text{card}(A)$.

(b) If $\text{card}(A) = \text{card}(B)$, then $\text{card}(B) = \text{card}(A)$.

(c) If $\text{card}(A) = \text{card}(B)$ and $\text{card}(B) = \text{card}(C)$, then $\text{card}(A) = \text{card}(C)$.

In light of the previous theorem, the next result should not be surprising.

Corollary 9.11. If X is a set, then "has the same cardinality as" is an equivalence relation on $\mathcal{P}(X)$.

Theorem 9.12. Let A, B, C, and D be sets such that $\text{card}(A) = \text{card}(C)$ and $\text{card}(B) = \text{card}(D)$.

(a) If A and B are disjoint and C and D are disjoint, then $\text{card}(A \cup B) = \text{card}(C \cup D)$.

(b) $\text{card}(A \times B) = \text{card}(C \times D)$.

Given two finite sets, it makes sense to say that one set is "larger than" another provided one set contains more elements than the other. We would like to generalize this idea to handle both finite and infinite sets.

Definition 9.13. Let A and B be sets. If there is an injective function from A to B, then we say that the **cardinality of** A **is less than or equal to the cardinality of** B. In this case, we write $\boxed{\text{card}(A) \leq \text{card}(B)}$.

Theorem 9.14. Let A, B, and C be sets.

(a) If $A \subseteq B$, then $\text{card}(A) \leq \text{card}(B)$.

(b) If $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(B) \leq \text{card}(C)$, then $\text{card}(A) \leq \text{card}(C)$.

(c) If $C \subseteq A$ while $\text{card}(B) = \text{card}(C)$, then $\text{card}(B) \leq \text{card}(A)$.

It might be tempting to think that the existence of injective function from a set A to a set B that is *not* surjective would verify that $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$. While this is true for finite sets, it is not true for infinite sets as the next problem asks you to verify.

Problem 9.15. Provide an example of sets A and B such that $\text{card}(A) = \text{card}(B)$ despite the fact that there exists an injective function from A to B that is not surjective.

Definition 9.16. Let A and B be sets. We write $\boxed{\text{card}(A) < \text{card}(B)}$ if $\text{card}(A) \leq \text{card}(B)$ and $\text{card}(A) \neq \text{card}(B)$.

As a reminder, the statements $\text{card}(A) = \text{card}(B)$ and $\text{card}(A) \leq \text{card}(B)$ are symbolic ways of asserting the existence of certain types of functions from A to B. When we write $\text{card}(A) < \text{card}(B)$, we are saying something much stronger than "There exists a function $f : A \to B$ that is injective but not surjective." The statement $\text{card}(A) < \text{card}(B)$ is asserting that *every* injective function from A to B is not surjective. In general, it is difficult to prove statements like $\text{card}(A) \neq \text{card}(B)$ or $\text{card}(A) < \text{card}(B)$.

> You will become clever through your mistakes.
>
> German Proverb

9.2 Finite Sets

In the previous section, we used the phrase "finite set" without formally defining it. Let's be a bit more precise. The following shorthand comes in handy.

Definition 9.17. For each $n \in \mathbb{N}$, define $\boxed{[n] := \{1, 2, \ldots, n\}}$.

For example, $[5] = \{1, 2, 3, 4, 5\}$. Notice that our notation looks just like the notation we used for equivalence classes. However, despite the similar notation, these concepts are unrelated. We will have to rely on context to keep them straight.

The next definition should coincide with your intuition about what it means for a set to be finite.

Definition 9.18. A set A is **finite** if $A = \emptyset$ or $\text{card}(A) = \text{card}([n])$ for some $n \in \mathbb{N}$. If $A = \emptyset$, then we say that A has **cardinality** 0 and if $\text{card}(A) = \text{card}([n])$, then we say that A has **cardinality** n.

Let's prove a few results about finite sets. When proving the following theorems, do not forget to consider the empty set.

Theorem 9.19. If A is finite and $\text{card}(A) = \text{card}(B)$, then B is finite.

Theorem 9.20. If A has cardinality $n \in \mathbb{N} \cup \{0\}$ and $x \notin A$, then $A \cup \{x\}$ is finite and has cardinality $n + 1$.

Consider using induction when proving the next theorem.

Theorem 9.21. For every $n \in \mathbb{N}$, every subset of $[n]$ is finite.

Theorem 9.20 shows that adding a single element to a finite set increases the cardinality by 1. As you would expect, removing one element from a finite set decreases the cardinality by 1.

Theorem 9.22. If A has cardinality $n \in \mathbb{N}$, then for all $x \in A$, $A \setminus \{x\}$ is finite and has cardinality $n - 1$.

The next result tells us that the cardinality of a proper subset of a finite set is never the same as the cardinality of the original set. It turns out that this theorem does not hold for infinite sets.

Theorem 9.23. Every subset of a finite set is finite. In particular, if A is a finite set, then $\text{card}(B) < \text{card}(A)$ for all proper subsets B of A.

Induction is a sensible approach to proving the next two theorems.

Theorem 9.24. If $A_1, A_2, \ldots, A_k$ is a finite collection of finite sets, then $\bigcup_{i=1}^{k} A_i$ is finite.

The next theorem, called the **Pigeonhole Principle**, is surprisingly useful. It puts restrictions on when we may have an injective function. The name of the theorem is inspired by the following idea: If n pigeons wish to roost in a house with k pigeonholes and $n > k$, then it must be the case that at least one hole contains more than one pigeon. Note that 2 is the smallest value of n that makes sense in the hypothesis below.

Theorem 9.25 (Pigeonhole Principle)**.** If $n, k \in \mathbb{N}$ and $f : [n] \to [k]$ with $n > k$, then f is not injective.

> God created infinity, and man, unable to understand infinity, had to invent finite sets.
>
> Gian-Carlo Rota, mathematician & philosopher

9.3 Infinite Sets

In the previous section, we explored finite sets. Now, let's tinker with infinite sets.

Definition 9.26. A set A is **infinite** if A is not finite.

Let's see if we can utilize this definition to prove that the set of natural numbers is infinite. For sake of a contradiction, assume otherwise. Then there exists $n \in \mathbb{N}$ such that $\text{card}([n]) = \text{card}(\mathbb{N})$, which implies that there exists a bijection $f : [n] \to \mathbb{N}$. What can you say about the number $m := \max(f(1), f(2), \ldots, f(n)) + 1$?

Theorem 9.27. The set $\mathbb{N}$ of natural numbers is infinite.

The next theorem is analogous to Theorem 9.19, but for infinite sets. To prove this theorem, try a proof by contradiction. You should end up composing two bijections, say $f : A \to B$ and $g : B \to [n]$ for some $n \in \mathbb{N}$. As we shall see later, the converse of this theorem is not true in general.

Theorem 9.28. If A is infinite and $\text{card}(A) = \text{card}(B)$, then B is infinite.

Problem 9.29. Quickly verify that the following sets are infinite by appealing to Theorem 9.27, Theorem 9.28, or Problem 9.6.

(a) The set of odd natural numbers

(b) The set of even natural numbers

(c) $\mathbb{Z}$

(d) $R = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$

(e) $\mathbb{N} \times \{a\}$

Notice that Definition 9.26 tells us what infinite sets are not, but it doesn't really tell us what they are. In light of Theorem 9.27, one way of thinking about infinite sets is as follows. Suppose A is some nonempty set. Let's select a random element from A and set it aside. We will call this element the "first" element. Then we select one of the remaining elements and set is aside, as well. This is the "second" element. Imagine we continue this way, choosing a "third" element, and "fourth" element, etc. If the set is infinite, we should never run out of elements to select. Otherwise, we would create a bijection with $[n]$ for some $n \in \mathbb{N}$.

The next problem, sometimes referred to as the Hilbert Hotel, named after the German mathematician David Hilbert (1862–1942), illustrates another way of thinking about infinite sets.

Problem 9.30. The Infinite Hotel has rooms numbered $1, 2, 3, 4, \ldots$. Every room in the Infinite Hotel is currently occupied.

(i) Is it possible to make room for one more guest (assuming they want a room all to themselves)?

(ii) An infinite number of new guests, say $g_1, g_2, g_3, \ldots$, show up in the lobby and each demands a room. Is it possible to make room for all the new guests even if the hotel is already full?

The previous problem verifies that there exists a proper subset of the natural numbers that is in bijection with the natural numbers themselves. We also witnessed this in Parts (a) and (b) of Problem 9.29. Notice that Theorem 9.23 forbids this type of behavior for finite sets. It turns out that this phenomenon is true for all infinite sets. The next theorem verifies that that the two viewpoints of infinite sets discussed above are valid. To prove this theorem, we need to prove that the three statements are equivalent. One possible approach is to prove (i) if and only if (ii) and (ii) if and only if (iii). For (i) implies (ii), construct f recursively. For (ii) implies (i), try a proof by contradiction. For (ii) implies (iii), let $B = A \setminus \{f(1), f(2), \ldots\}$ and show that A can be put in bijection with $B \cup \{f(2), f(3), \ldots\}$. Lastly, for (iii) implies (ii), suppose $g : A \to C$ is a bijection for some proper subset C of A. Let $a \in A \setminus C$. Define $f : \mathbb{N} \to A$ via $f(n) = g^n(a)$, where g^n means compose g with itself n times.

Theorem 9.31. The following statements are equivalent.

(i) The set A is infinite.

(ii) There exists an injective function $f : \mathbb{N} \to A$.

(iii) The set A can be put in bijection with a proper subset of A (i.e., there exists a proper subset B of A such that $\text{card}(B) = \text{card}(A)$).

It is worth mentioning that for the previous theorem, (iii) implies (i) follows immediately from the contrapositive of Theorem 9.23. When proving (i) implies (ii) in the previous theorem, did you apply the Axiom of Choice? If so, where?

Corollary 9.32. A set is infinite if and only if it has an infinite subset.

Corollary 9.33. If A is an infinite set, then $\text{card}(\mathbb{N}) \leq \text{card}(A)$.

Problem 9.34. Find a new proof of Theorem 9.27 that uses (iii) implies (i) from Theorem 9.31.

Problem 9.35. Quickly verify that the following sets are infinite by appealing to either Theorem 9.31 (use (ii) implies (i)) or Corollary 9.32.

(a) Set of odd natural numbers

(b) Set of even natural numbers

(c) $\mathbb{Z}$

(d) $\mathbb{N} \times \mathbb{N}$

(e) $\mathbb{Q}$

(f) $\mathbb{R}$

(g) Set of perfect squares in $\mathbb{N}$

(h) $(0, 1)$

(i) $\mathbb{C} := \{a + bi \mid a, b \in \mathbb{R}\}$

> An ounce of practice is worth more than tons of preaching.
>
> Mahatma Gandhi, political activist

9.4 Countable Sets

Recall that if $A = \emptyset$, then we say that A has cardinality 0. Also, if $\text{card}(A) = \text{card}([n])$ for $n \in \mathbb{N}$, then we say that A has cardinality n. We have a special way of describing sets that are in bijection with the natural numbers.

Definition 9.36. If A is a set such that $\text{card}(A) = \text{card}(\mathbb{N})$, then we say that A is **denumerable** and has **cardinality** $\aleph_0$ (read "aleph naught").

Notice if a set A has cardinality $1, 2, \ldots$, or $\aleph_0$, we can label the elements in A as "first", "second", and so on. That is, we can "count" the elements in these situations. Certainly, if a set has cardinality 0, counting is not an issue. This idea leads to the following definition.

Definition 9.37. A set A is called **countable** if A is finite or denumerable. A set is called **uncountable** if it is not countable.

Problem 9.38. Quickly justify that each of the following sets is countable. Feel free to appeal to previous problems. Which sets are denumerable?

(a) $\{a, b, c\}$

(b) Set of odd natural numbers

(c) Set of even natural numbers

(d) $\{\frac{1}{2^n} \mid n \in \mathbb{N}\}$

(e) Set of perfect squares in $\mathbb{N}$

(f) $\mathbb{Z}$

(g) $\mathbb{N} \times \{a\}$

Utilize Theorem 9.31 or Corollary 9.33 when proving the next result.

Theorem 9.39. Every infinite set contains a denumerable subset.

Theorem 9.40. Let A and B be sets such that A is countable. If $f : A \to B$ is a bijection, then B is countable.

For the next proof, consider the cases when A is finite versus infinite. The contrapositive of Corollary 9.32 should be useful for the case when A is finite.

Theorem 9.41. Every subset of a countable set is countable.

Theorem 9.42. A set is countable if and only if it has the same cardinality of some subset of the natural numbers.

Theorem 9.43. If $f : \mathbb{N} \to A$ is a surjective function, then A is countable.

Loosely speaking, the next theorem tells us that we can arrange all of the rational numbers then count them "first", "second", "third", etc. Given the fact that between any two distinct rational numbers on the number line, there are an infinite number of other rational numbers (justified by taking repeated midpoints), this may seem counterintuitive.

Here is one possible approach for proving the next theorem. Make a table with column headings $0, 1, -1, 2, -2, \ldots$ and row headings $1, 2, 3, 4, 5, \ldots$. If a column has heading m and a row has heading n, then the entry in the table corresponds to the fraction m/n. Find a way to zig-zag through the table making sure to hit every entry in the table (not including column and row headings) exactly once. This justifies that there is a bijection between $\mathbb{N}$ and the entries in the table. Do you see why? But now notice that every rational number appears an infinite number of times in the table. Resolve this by appealing to Theorem 9.41.

Theorem 9.44. The set of rational numbers $\mathbb{Q}$ is countable.

Theorem 9.45. If A and B are countable sets, then $A \cup B$ is countable.

We would like to prove a stronger result than the previous theorem. To do so, we need an intermediate result.

Theorem 9.46. Let $\{A_n\}_{n=1}^{\infty}$ be a collection of sets. Define $B_1 := A_1$ and for each natural number $n > 1$, define

$$B_n := A_n \setminus \bigcup_{i=1}^{n-1} A_i.$$

Then we we have the following:

(a) The collection $\{B_n\}_{n=1}^{\infty}$ is pairwise disjoint.

(b) $\displaystyle\bigcup_{n=1}^{\infty} A_n = \bigcup_{n=1}^{\infty} B_n$.

The next theorem states that the union of a countable collection of countable sets is countable. To prove this, consider two cases:

1. The collection of sets is finite.

2. The collection of sets is infinite.

To handle the first case, use induction together with Theorem 9.45. The second case is substantially more challenging. First, use Theorem 9.46 to construct a collection $\{B_n\}$ of pairwise disjoint sets whose union is equal to the union of the original collection. Since each B_n is a subset of one of the sets in the original collection and each of these sets is countable, each B_n is also countable by Theorem 9.41. This implies that for each n, we can write $B_n = \{b_{n,1}, b_{n,2}, b_{n,3}, \ldots\}$, where the set may be finite or infinite. From here, we outline two different approaches for continuing. One approach is to construct a bijection from $\mathbb{N}$ to $\bigcup_{n=1}^{\infty} B_n$ using Figure 9.2 as inspiration. One thing you will need to address is what to do when a set in the collection $\{B_n\}$ is finite. For the second approach, define $f : \bigcup_{n=1}^{\infty} B_n \to \mathbb{N}$ via $f(b_{n,m}) = 2^n 3^m$, verify that this function is injective, and then appeal to Theorem 9.41. Try using both of these approaches when tackling the proof of the following theorem.

Theorem 9.47. Let Δ be equal to either $\mathbb{N}$ or $[k]$ for some $k \in \mathbb{N}$. If $\{A_n\}_{n\in\Delta}$ is a countable collection of sets such that each A_n is countable, then $\bigcup_{n\in\Delta} A_n$ is countable.

Do you use the Axiom of Choice when proving the previous theorem? If so, where?

Theorem 9.48. If A and B are countable sets, then $A \times B$ is countable.

Theorem 9.49. The set of all finite sequences of 0's and 1's (e.g., 0110010 is a finite sequence consisting of 0's and 1') is countable.

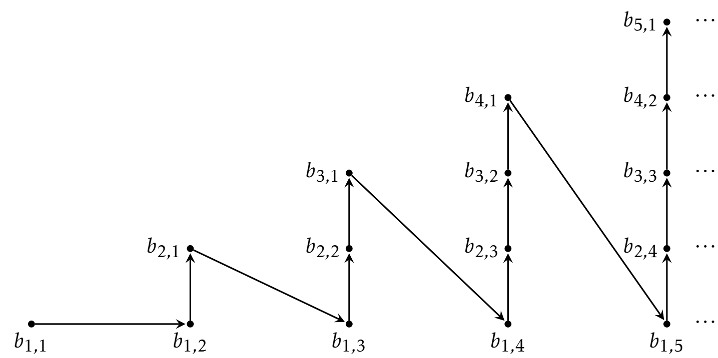

Figure 9.2: Inspiration for one possible approach to proving Theorem 9.47.

Theorem 9.50. The collection of all finite subsets of a countable set is countable.

> Vulnerability is not winning or losing; it's having the courage to show up and be seen when we have no control over the outcome.
>
> Brené Brown, storyteller & author

9.5 Uncountable Sets

Recall from Definition 9.37 that a set A is uncountable if A is not countable. Since all finite sets are countable, the only way a set could be uncountable is if it is infinite. It follows that a set A is uncountable if and only if there is never a bijection between $\mathbb{N}$ and A. It is not clear that uncountable sets even exist! It turns out that uncountable sets do exist and in this section, we will discover a few of them.

Our first task is to prove that the interval $(0,1)$ is uncountable. By Problem 9.35(h), we know that $(0,1)$ is an infinite set, so it is at least plausible that $(0,1)$ is uncountable. The following problem outlines the proof of Theorem 9.52. Our approach is often referred to as **Cantor's Diagonalization Argument**, named after German mathematician Georg Cantor (1845–1918).

Before we get started, recall that every number in $(0,1)$ can be written in decimal form. However, there may be more than one way to write a given number in decimal form. For example, 0.2 equals $0.1\overline{99}$. A number $0.a_1a_2a_3\ldots$ in $(0,1)$ is said to be in **standard decimal form** if there is no k such that for all $i > k$, $a_i = 9$. That is, a number is in standard decimal form if and only if its decimal expansion does not end with a repeating sequence of 9's. For example, 0.2 is in standard decimal form while $0.1\overline{99}$ is not, even though both represent the same number. It turns out that every real number can be expressed uniquely in standard decimal form. We will take this fact for granted.

Problem 9.51. For sake of a contradiction, assume the interval $(0,1)$ is countable. Then there exists a bijection $f : \mathbb{N} \to (0,1)$. For each $n \in \mathbb{N}$, its image under f is some number in $(0,1)$. Write $f(n) = 0.a_{1n}a_{2n}a_{3n}\ldots$, where a_{1n} is the first digit in the standard decimal form for the image of n, a_{2n} is the second digit, and so on. If $f(n)$ terminates after k digits, then our convention will be to continue the decimal expansion with 0's. Now, define $b = 0.b_1b_2b_3\ldots$, where

$$b_i = \begin{cases} 2, & \text{if } a_{ii} \neq 2 \\ 3, & \text{if } a_{ii} = 2. \end{cases}$$

(a) Prove that the decimal expansion that defines b above is in standard decimal form.

(b) Prove that for all $n \in \mathbb{N}$, $f(n) \neq b$.

(c) Explain why f cannot be surjective and why this is a contradiction.

You just proved that the interval $(0,1)$ cannot be countable!

The previous problem proves following theorem.

Theorem 9.52. The open interval $(0,1)$ is uncountable.

Loosely speaking, what Theorem 9.52 says is that the open interval $(0,1)$ is "bigger" in terms of the number of elements it contains than the natural numbers and even the rational numbers. This shows that there are infinite sets of different sizes! We now know there is at least one uncountable set, namely the interval $(0,1)$. The next three results are useful for finding other uncountable sets. For the first theorem, try a proof by contradiction and take a look at Theorem 9.41.

Theorem 9.53. If A and B are sets such that $A \subseteq B$ and A is uncountable, then B is uncountable.

Corollary 9.54. If A and B are sets such that A is uncountable and B is countable, then $A \setminus B$ is uncountable.

Theorem 9.55. If $f : A \to B$ is an injective function and A is uncountable, then B is uncountable.

Since the interval $(0,1)$ is uncountable and $(0,1) \subseteq \mathbb{R}$, it follows immediately from Theorem 9.53 that $\mathbb{R}$ is also uncountable. The next theorem tells that $(0,1)$ and $\mathbb{R}$ actually have the same cardinality! To prove this, consider the function $f : (0,1) \to \mathbb{R}$ defined via $f(x) = \tan(\pi x - \frac{\pi}{2})$.

Theorem 9.56. The set of real numbers is uncountable. In particular, $\text{card}((0,1)) = \text{card}(\mathbb{R})$.

The **continuum hypothesis**—originally proposed by Cantor in 1878—states that there is no set whose cardinality is strictly between that of the natural numbers and the real numbers. Cantor unsuccessfully attempted to prove the continuum hypothesis for several years. It follows from the work of Paul Cohen (1934–2007) and Kurt Gödel (1906–1978) that the continuum hypothesis and its negation are independent of the Zermelo-Fraenkel axioms of set theory (briefly discussed at the end of Section 3.2). That is, either the continuum hypothesis or its negation can be added as an axiom to ZFC set theory, with the resulting theory being consistent if and only if ZFC is consistent (i.e., no contradictions are produced). Nowadays, most set theorists believe that the continuum hypothesis *should* be false.

Theorem 9.57. If $a, b \in \mathbb{R}$ with $a < b$, then (a,b), $[a,b]$, $(a,b]$, and $[a,b)$ are all uncountable.

Theorem 9.58. The set of irrational numbers is uncountable.

Theorem 9.59. The set $\mathbb{C}$ of complex numbers is uncountable.

Problem 9.60. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If A and B are sets such that A is uncountable, then $A \cup B$ is uncountable.

(b) If A and B are sets such that A is uncountable, then $A \cap B$ is uncountable.

(c) If A and B are sets such that A is uncountable, then $A \times B$ is uncountable.

(d) If A and B are sets such that A is uncountable, then $A \setminus B$ is uncountable.

An approach similar to Cantor's Diagonalization Argument will be helpful when approaching the next problem.

Problem 9.61. Let S be the set of infinite sequences of 0's and 1's. Determine whether S is countable or uncountable and prove that your answer is correct.

Theorem 9.62. If S is the set from Problem 9.61, then $\text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(S)$.

Corollary 9.63. The power set of the natural numbers is uncountable.

Notice that $\mathbb{N}$ is countable while $\mathcal{P}(\mathbb{N})$ is uncountable. That is, the power set of the natural numbers has cardinality strictly larger than the natural numbers. We generalize this phenomenon in the next theorem.

According to Theorem 9.56 and Corollary 9.63, $\mathbb{R}$ and $\mathcal{P}(\mathbb{N})$ are both uncountable. In fact, $\text{card}(\mathcal{P}(\mathbb{N})) = \text{card}(\mathbb{R})$, which we state without proof. However, it turns out that the two uncountable sets may or may not have the same cardinality. Perhaps surprisingly, there are sets that are even "bigger" than the set of real numbers. The next theorem is named after Georg Cantor, who first stated and proved it at the end of the 19th century. The theorem states that given any set, we can always increase the cardinality by considering its power set. That is, if A is a set, $\mathcal{P}(A)$ has strictly greater cardinality than A itself. For finite sets, Cantor's theorem follows from Theorems 4.11 and 4.12 (both of which we proved via induction). Perhaps much more surprising is that Cantor discovered an elegant argument that is applicable to any set, whether finite or infinite. To prove Cantor's Theorem, first exhibit an injective function from A to $\mathcal{P}(A)$. This proves that $\text{card}(A) \leq card(\mathcal{P}(A))$. To show that $\text{card}(A) < \text{card}(\mathcal{P}(A))$, try a proof by contradiction. That is, assume there exists a bijective function $f : A \to \mathcal{P}(A))$. Derive a contradiction by considering the set $B = \{x \in A \mid x \notin f(x)\}$.

Theorem 9.64 (Cantor's Theorem)**.** If A is a set, then $\text{card}(A) < \text{card}(\mathcal{P}(A))$.

Recall that cardinality provides a way for talking about "how big" a set is. The fact that the natural numbers and the real numbers have different cardinality (one countable, the other uncountable), tells us that there are at least two different "sizes of infinity". By iteratively taking the power set of an infinite set and applying Cantor's Theorem we obtain an endless hierarchy of cardinalities, each strictly larger than the one before it. Colloquially, this implies that there are "infinitely many sizes of infinity" and there is "no largest infinity".

> If you want to sharpen a sword, you have to remove a little metal.
>
> Author Unknown