

An Introduction to Proof via Inquiry-Based Learning

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This book is intended to be a task sequence for an introduction to proof course that utilizes an inquiry-based learning (IBL) approach. You can find the most up-to-date version of this textbook on GitHub:

<http://dcernst.github.io/IBL-IntroToProof/>

I would be thrilled if you used this textbook and improved it. If you make any modifications, you can either make a pull request on GitHub or submit the improvements via email. You are also welcome to fork the source and modify the text for your purposes as long as you maintain the license below.

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