I write one page of masterpiece to ninety-one pages of shit.

Ernest Hemingway, novelist & journalist

Chapter 8

Functions

In this chapter, we will introduce the concept of function as a special type of relation. Our definition should agree with any previous definition of function that you may have learned. We will also study various properties that a function may or may not possess.

8.1 Introduction to Functions

Up until this point, you may have only encountered functions as an algebraic rule, e.g., $f(x) = x^2 - 1$, for transforming one real number into another. However, we can study functions in a much broader context. The basic building blocks of a function are a first set and a second set, say X and Y, and a "correspondence" that assigns *every* element of X to *exactly one* element of Y. Let's take a look at the actual definition.

Definition 8.1. Let X and Y be two nonempty sets. A **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x, y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\mathrm{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\mathrm{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\mathrm{Rng}(f) := \{y \in Y \mid \text{there exists } x \text{ such that } (x, y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via..." or "Define $f : X \to Y$ via...", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(a, b) \in f$ for a function f, we often write $\boxed{f(a) = b}$ and say that "f maps a to b" or "f of a equals b". In this case, a may be called an **input** of f and is the **preimage** of b under f while b is called an **output** of f and is the **image** of a under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

According to our definition, if $f : X \to Y$ is a function, then every element of the domain is utilized exactly once. However, there are no restrictions on whether an element of the codomain ever appears in the second coordinate of an ordered pair in the relation. Yet if an element of Y is in the range of f, it may appear in more than one ordered pair in the relation.

It follows immediately from the definition of function that two functions are equal if and only if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, functions f and g are equal if and only if $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, $\mathrm{Codom}(f) = \mathrm{Codom}(g)$, and $f(x) = g(x)$ for all $x \in X$.

Since functions are special types of relations, we can represent them using digraphs and graphs when practical. Digraphs for functions are often called **function** (or **mapping**) **diagrams**. When drawing function diagrams, it is standard practice to put the vertices for the domain on the left and the vertices for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 8.2. Let $X = \{a, b, c, d\}$ to $Y = \{1, 2, 3, 4\}$ and define the relation f from X to Y via

$$f = \{(a, 2), (b, 4), (c, 4), (d, 1)\}.$$

Since each element X appears exactly once as a first coordinate, f is a function with domain X and codomain Y (i.e., $f : X \to Y$). In this case, we see that $\mathrm{Rng}(f) = \{1, 2, 4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." The function diagram for f is depicted in Figure 8.1.

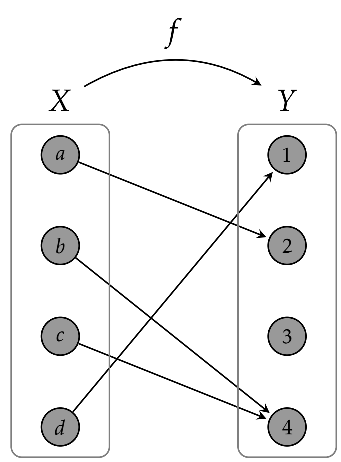


Figure 8.1: Function diagram for a function from $X = \{a, b, c, d, \}$ to $Y = \{1, 2, 3, 4\}$.

Problem 8.3. Determine whether each of the relations defined in the following examples and problems is a function.

(a) Example 7.3 (see Figure 7.1)

(b) Example 7.14 (see Figure 7.3)

(c) Problem 7.15

(d) Problem 7.21

Problem 8.4. Let $X = \{\circ, \square, \triangle, \smiley\}$ and $Y = \{a, b, c, d, e\}$. For each of the following relations, draw the corresponding digraph and determine whether the relation represents a function from X to Y, Y to X, X to X, or does not represent a function. If the relation is a function, determine the domain, codomain, and range.

(a) $f = \{(\circ, a), (\square, b), (\triangle, c), (\smiley, d)\}$

(b) $g = \{(\circ, a), (\square, b), (\triangle, c), (\smiley, c)\}$

(c) $h = \{(\circ, a), (\square, b), (\triangle, c), (\circ, d)\}$

(d) $k = \{(\circ, a), (\square, b), (\triangle, c), (\smiley, d), (\square, e)\}$

(e) $l = \{(\circ, e), (\square, e), (\triangle, e), (\smiley, e)\}$

(f) $m = \{(\circ, a), (\triangle, b), (\smiley, c)\}$

(g) $i = \{(\circ, \circ), (\square, \square), (\triangle, \triangle), (\smiley, \smiley)\}$

(h) Define the relation happy from Y to X via $(y, \smiley) \in \text{happy}$ for all $y \in Y$.

(i) $\text{nugget} = \{(\circ, \circ), (\square, \square), (\triangle, \triangle), (\smiley, \square)\}$

The last two parts of the previous problem make it clear that functions may have names consisting of more than one letter. The function names sin, cos, log, and ln are instances of this that you have likely encountered in your previous experience in mathematics. One thing that you may have never noticed is the type of font that we use for function names. It is common to italicize generic function names like f but not common function names like sin. However, we always italicize the variables used to represent the input and output for a function. For example, consider the font types used in the expressions $\sin(x)$ and $\ln(a)$.

Problem 8.5. What properties does the digraph for a relation from X to Y need to have in order for it to represent a function?

Problem 8.6. In high school you may have been told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However, notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal.

Example 8.7. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Sometimes we rely on context to interpret the domain and codomain. For example, in a calculus class, when we describe a function in terms of a formula, we are implicitly assuming that the domain is the largest allowable subset of $\mathbb{R}$—sometimes called the **default domain**—that makes sense for the given formula while the codomain is $\mathbb{R}$.

Example 8.8. If we write $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$ without mentioning the domains, we would typically interpret these as the functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$, and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that are determined by their respective formulas.

Problem 8.9. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(g)$ is strictly smaller than $\mathrm{Codom}(g)$.

Problem 8.10. Let $f : X \to Y$ be a function and suppose that X and Y are finite sets with n and m elements, respectively, such that $n < m$. Is it possible for $\mathrm{Rng}(f) = \mathrm{Codom}(f)$? If so, provide an example. If this is not possible, explain why.

There are a few special functions that we should know the names of.

Definition 8.11. If X and Y are nonempty sets such that $X \subseteq Y$, then the function $\iota : X \to Y$ defined via $\iota(x) = x$ is called the **inclusion map from X into Y**.

Note that "ι" is the Greek letter "iota".

Problem 8.12. Let $X = \{a, b, c\}$ and $Y = \{a, b, c, d\}$. Draw the function diagram of the inclusion map from X into Y.

If the domain and codomain are equal, the inclusion map has a special name.

Definition 8.13. If X is a nonempty set, then the function $i_X : X \to X$ defined via $i_X(x) = x$ is called the **identity map** (or **identity function**) **on** X.

Example 8.14. The relation defined in Problem 8.4(g) is the identity map on $X = \{\circ, \square, \triangle, ☺\}$.

Problem 8.15. Draw a portion of the graph of the identity map on $\mathbb{R}$ as a subset of $\mathbb{R}^2$.

Definition 8.16. Any function $f : X \to Y$ defined via $f(x) = c$ for a fixed $c \in Y$ is called a **constant function**.

Example 8.17. The function defined in Problem 8.4(h) is an example of a constant function. Notice that we can succinctly describe this function using the formula $\mathrm{happy}(y) = ☺$.

Definition 8.18. A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition of the domain.

Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 8.19. We can express the function in Problem 8.4(i) as a piecewise function using the formula

$$\text{nugget}(x) = \begin{cases} x, & \text{if } x \text{ is a geometric shape,} \\ \square, & \text{otherwise.} \end{cases}$$

Example 8.20. The function $f : \mathbb{R} \to \mathbb{R}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \geq 0, \\ 17, & \text{if } -2 \leq x < 0, \\ -x, & \text{if } x < -2 \end{cases}$$

is an example of a piecewise-defined function.

Problem 8.21. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{|x|}{x}$. Express f as a piecewise function.

It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 8.22. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1,2,3\} \to \{1,2,3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1,2,3\}$ and $A_2 = \{3,4,5\}$. Define $h : A_1 \cup A_2 \to \{1,2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication. This issue was lurking behind the scenes in the statement of Theorem 7.94. In particular, the expressions

$$[a_1]_n + [a_2]_n + \cdots + [a_k]_n$$

and
$$[a_1]_n[a_2]_n\cdots[a_k]_n$$
are well defined in $\mathbb{Z}/n\mathbb{Z}$ in light of Theorems 7.92(b) and 7.93(b).

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of representatives of equivalence classes, or more generally in terms of how an element of the domain is written. For example, the descriptions given in Parts (c) and (d) of Problem 8.22 are not well defined. To show that a potentially ambiguous description for a function $f : X \to Y$ is well defined prove that if a and b are two representations for the same element in X, then $f(a) = f(b)$.

Problem 8.23. For each of the following, determine whether the description determines a well-defined function.

(a) Define $f : \mathbb{Z}/5\mathbb{Z} \to \mathbb{N}$ via
$$f([a]_5) = \begin{cases} 0, & \text{if } a \text{ is even} \\ 1, & \text{if } a \text{ is odd.} \end{cases}$$

(b) Define $g : \mathbb{Z}/6\mathbb{Z} \to \mathbb{N}$ via
$$g([a]_6) = \begin{cases} 0, & \text{if } a \text{ is even} \\ 1, & \text{if } a \text{ is odd.} \end{cases}$$

(c) Define $m : \mathbb{Z}/8\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ via $m([x]_8) = [6x]_{10}$.

(d) Define $h : \mathbb{Z}/10\mathbb{Z} \to \mathbb{Z}/10\mathbb{Z}$ via $h([x]_{10}) = [6x]_{10}$.

(e) Define $k : \mathbb{Z}/43\mathbb{Z} \to \mathbb{Z}/43\mathbb{Z}$ via $k([x]_{43}) = [11x-5]_{43}$.

(f) Define $\ell : \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/15\mathbb{Z}$ via $\ell([x]_{15}) = [5x-11]_{15}$.

Problem 8.24. Let $k, n \in \mathbb{N}$ and $m \in \mathbb{Z}$. Under what conditions will $f_m : \mathbb{Z}/n\mathbb{Z} \to \mathbb{Z}/k\mathbb{Z}$ given by $f_m([x]_n) = [mx]_k$ be a well-defined function? Prove your claim.

> Don't let anyone rob you of your imagination, your creativity, or your curiosity. It's your place in the world; it's your life. Go on and do all you can with it, and make it the life you want to live.
>
> Mae Jemison, NASA astronaut

8.2 Injective and Surjective Functions

We now turn our attention to some important properties that a function may or may not possess. Recall that if f is a function, then every element in its domain is mapped to a unique element in the range. However, there are no restrictions on whether more than one element of the domain is mapped to the same element in the range. If each element in the range has a unique element in the domain mapping to it, then we say that the function is injective. Moreover, the range of a function is not required to be all of the codomain. If every element of the codomain has at least one element in the domain that maps to it, then we say that the function is surjective. Let's make these definitions a bit more precise.

Definition 8.25. Let $f : X \to Y$ be a function.

(a) The function f is said to be **injective** (or **one-to-one**) if for all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.

(b) The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.

(c) If f is both injective and surjective, we say that f is **bijective**.

Problem 8.26. Compare and contrast the following statements. Do they mean the same thing?

(a) For all $x \in X$, there exists a unique $y \in Y$ such that $f(x) = y$.

(b) For all $y \in \mathrm{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.

Problem 8.27. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 8.28. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is a bijection.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither injective nor surjective.

(e) A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is injective.

Problem 8.29. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is ________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true.

Problem 8.30. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is ________ if and only if every horizontal line hits the graph of f *at least once*.

Explain why this statement is true.

Problem 8.31. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is ________ if and only if every horizontal line hits the graph of f *exactly once*.

Explain why this statement is true.

How do we prove that a function f is injective? We would need to show that every element in the range has a unique element from the domain that maps to it. First, notice that each element in the range can be written as $f(x)$ for at least one x in the domain. To argue that each such element in domain is unique, we can suppose $f(x_1) = f(x_2)$ for arbitrary x_1 and x_2 in the domain and then work to show that $x_1 = x_2$. It is important to point out that when we suppose $f(x_1) = f(x_2)$ for some x_1 and x_2, we are not assuming that x_1 and x_2 are different. In general, when we write "Let $x_1, x_2 \in X \ldots$", we are leaving open the possibility that x_1 and x_2 are actually the same element. One could approach proving that a function is injective by utilizing a proof by contradiction, but this is not usually necessary.

Skeleton Proof 8.32 (Proof that a function is injective)**.** Here is the general structure for proving that a function is injective.

> *Proof.* Assume $f : X \to Y$ is a function defined by (or satisfying)... *[Use the given definition (or describe the given property) of f].* Let $x_1, x_2 \in X$ and suppose $f(x_1) = f(x_2)$.
>
> ... *[Use the definition (or property) of f to verify that $x_1 = x_2$]* ...
>
> Therefore, the function f is injective. □

How do we prove that a function f is surjective? We would need to argue that every element in the codomain is also in the range. Sometimes, the proof that a particular function is surjective is extremely short, so do not second guess yourself if you find yourself in this situation.

Skeleton Proof 8.33 (Proof that a function is surjective)**.** Here is the general structure for proving that a function is surjective.

> *Proof.* Assume $f : X \to Y$ is a function defined by (or satisfying)... *[Use the given definition (or describe the given property) of f].* Let $y \in Y$.
>
> ... *[Use the definition (or property) of f to find some $x \in X$ such that $f(x) = y$]* ...
>
> Therefore, the function f is surjective. □

Problem 8.34. Determine whether each of the following functions is injective, surjective, both, or neither. In each case, you should provide a proof or a counterexample as appropriate.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0,\infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x,y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n,n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n+1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

(i) The function h defined in Problem 8.23(d).

(j) The function k defined in Problem 8.23(e).

(k) The function ℓ defined in Problem 8.23(f).

Problem 8.35. Suppose X and Y are nonempty sets with m and n elements, respectively, where $m \leq n$. How many injections are there from X to Y?

Problem 8.36. Compare and contrast the definition of "function" with the definition of "injective function". Consider the vertical line test and horizontal line test in your discussion. Moreover, attempt to capture what it means for a relation to not be a function and for a function to not be an injection by drawing portions of a digraph.

The next two theorems should not come as as surprise.

Theorem 8.37. The inclusion map $\iota : X \to Y$ for $X \subseteq Y$ is an injection.

Theorem 8.38. The identity function $i_X : X \to X$ is a bijection.

Problem 8.39. Let A and B be nonempty sets and let S be a nonempty subset of $A \times B$. Define $\pi_1 : S \to A$ and $\pi_2 : S \to B$ via $\pi_1(a,b) = a$ and $\pi_2(a,b) = b$. We call π_1 and π_2 the **projections** of S onto A and B, respectively.

(a) Provide examples to show that π_1 does not need to be injective nor surjective.

(b) Suppose that S is also a function. Is π_1 injective? Is π_1 surjective? How about π_2?

The next theorem says that if we have an equivalence relation on a nonempty set, the mapping that assigns each element to its respective equivalence class is a surjective function.

Theorem 8.40. If $\sim$ is an equivalence relation on a nonempty set A, then the function $f : A \to A/{\sim}$ defined via $f(x) = [x]$ is a surjection.

The function from the previous theorem is sometimes called the **canonical projection map** induced by $\sim$.

Problem 8.41. Under what circumstances would the function from the previous theorem also be injective?

Let's explore whether we can weaken the hypotheses of Theorem 8.40.

Problem 8.42. Let R be a relation on a nonempty set A.

(a) What conditions on R must hold in order for $f : A \to \mathrm{Rel}(R)$ defined via $f(a) = \mathrm{rel}(a)$ to be a function?

(b) What additional conditions, if any, must hold on R in order for f to be a surjective function?

Problem 8.43. Let A be a nonempty set.

(a) Suppose R is an equivalence relation on A. Under what conditions is R a function from A to A?

(b) Suppose $f : A \to A$ is a function. Under what conditions is f an equivalence relation on A?

Given any function, we can define an equivalence relation on its domain, where the equivalence classes correspond to the elements that map to the same element of the range.

Theorem 8.44. Let $f : X \to Y$ be a function and define $\sim$ on X via $a \sim b$ if $f(a) = f(b)$. Then $\sim$ is an equivalence relation on X.

It follows immediately from Theorem 7.59 that the equivalence classes induced by the equivalence relation in Theorem 8.44 partition the domain of a function.

Problem 8.45. For each of the following, identify the equivalence classes induced by the relation from Theorem 8.44 for the given function.

(a) The function f defined in Example 8.2.

(b) The function c defined in Problem 8.34(e).

If f is a function, the equivalence relation in Theorem 8.44 allows us to construct a bijective function whose domain is the set of equivalence classes and whose codomain coincides with the range of f. This is an important idea that manifests itself in many areas of mathematics. One such instance is the First Isomorphism Theorem for Groups, which is a fundamental theorem in a branch of mathematics called group theory. When proving the following theorem, the first thing you should do is verify that the description for $\overline{f}$ is well defined.

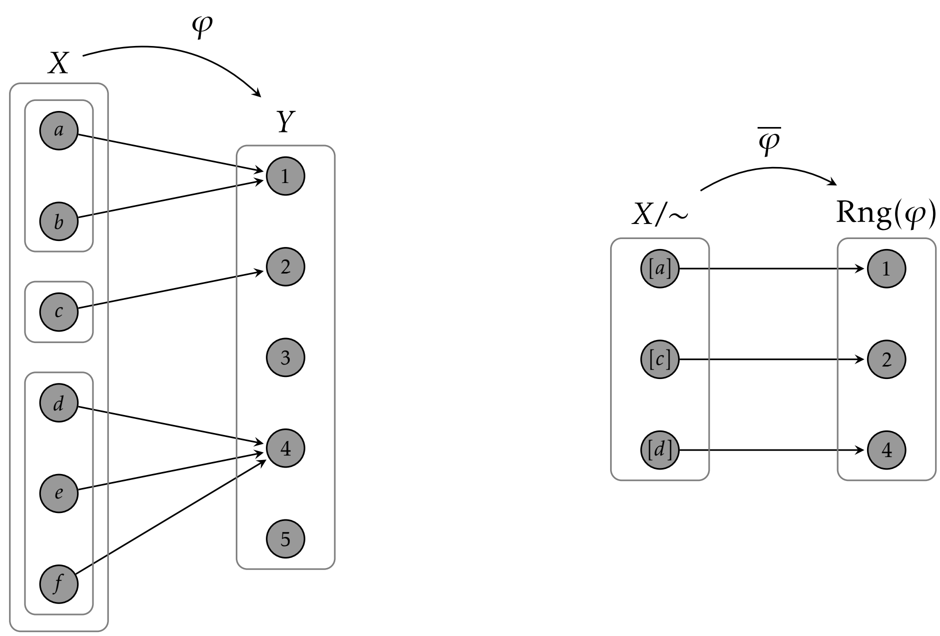
Theorem 8.46. Let $f : X \to Y$ be a function and define $\sim$ on X as in Theorem 8.44. Then the function $\overline{f} : X/\!\sim\, \to \operatorname{Rng}(f)$ defined via $\overline{f}([a]) = f(a)$ is a bijection.

Here is an analogy for helping understand the content of Theorem 8.46. Suppose we have a collection airplanes filled with passengers and a collection of potential destination cities such that at most one airplane may land at each city. The function f indicates which city each passenger lands at while the function $\overline{f}$ indicates which city each airplane lands at. Moreover, the codomain for the function $\overline{f}$ consists only of the cities that an airplane lands at.

Example 8.47. Let $X = \{a, b, c, d, e, f\}$ and $Y = \{1, 2, 3, 4, 5\}$ and define $\varphi : X \to Y$ via

$$\varphi = \{(a, 1), (b, 1), (c, 2), (d, 4), (e, 4), (f, 4)\}.$$

The function diagram for φ is given in Figure 8.2(a), where we have highlighted the elements of the domain that map to the same element in the range by enclosing them in additional boxes. We see that $\operatorname{Rng}(\varphi) = \{1, 2, 4\}$. The function diagram for the induced map $\overline{\varphi}$ that is depicted in Figure 8.2(b) makes it clear that $\overline{\varphi}$ is a bijection. Note that since $\varphi(a) = \varphi(b)$ and $\varphi(d) = \varphi(e) = \varphi(f)$, it must be the case that $[a] = [b]$ and $[d] = [e] = [f]$ according to Theorem 7.42. Thus, the vertices labeled as $[a]$ and $[d]$ in Figure 8.2(b) could have also been labeled as $[b]$ and $[c]$ or $[d]$, respectively. In terms of our passengers and airplanes analogy, $X = \{a, b, c, d, e, f\}$ is the set of passengers, $Y = \{1, 2, 3, 4, 5\}$ is the set of potential destination cities, $X/\!\sim\, = \{[a], [c], [d]\}$ is the set of airplanes, and $\operatorname{Rng}(\varphi) = \{1, 2, 4\}$ is the set of cities that airplanes land at. The equivalence class $[a]$ is the airplane containing the passenger a, and since a and b are on the same plane, $[b]$ is also the plane containing the passenger a.



(a) (b)

Figure 8.2: Example of a visual representation of Theorem 8.46.

Problem 8.48. Consider the equivalence classes you identified in Parts (a) and (b) of Problem 8.45.

(a) Draw the function diagram for the function $\overline{f}$ as defined in Theorem 8.46, where f is the function defined in Example 8.2.

(b) Describe the function $\bar{c}$ as defined in Theorem 8.46, where c is the function defined in Problem 8.34(e).

While perhaps not surprising, Problem 8.48(b) tells us that there is a one-to-one correspondence between circles centered at the origin and real numbers.

Problem 8.49. Let $Y = \{0, 1, 2, 3\}$ and define the function $f : \mathbb{Z} \to Y$ such that $f(n)$ equals the unique remainder obtained after dividing n by 4. For example, $f(11) = 3$ since $11 = 4 \cdot 2 + 3$ according to the Division Algorithm (Theorem 6.7). This function is sometimes written as $f(n) = n \pmod 4$, where it is understood that we restrict the output to $\{0, 1, 2, 3\}$. It is clear that f is surjective since 0, 1, 2, and 3 are mapped to 0, 1, 2, and 3, respectively. Figure 8.3 depicts a portion of the function diagram for f, where we have drawn the diagram from the top down instead of left to right.

(a) Describe the equivalence classes induced by the relation given in Theorem 8.44.

(b) What familiar set is $\mathbb{Z}/\!\sim$ equal to?

(c) Draw the function diagram for the function $\overline{f}$ as defined in Theorem 8.46.

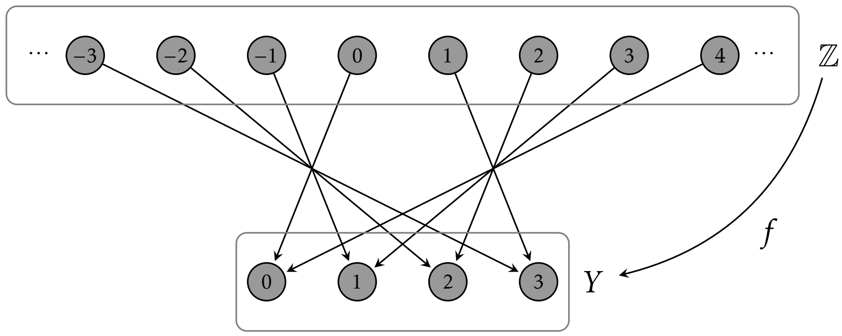


Figure 8.3: Function diagram for the function described in Problem 8.49.

(d) The function diagram in Figure 8.3 is a bit hard to interpret due to the ordering of the elements in the domain. Can you find a better way to lay out the vertices in the domain that makes the function f easier to interpret?

Problem 8.50. Consider the function h defined in Problem 8.23(d).

(a) Draw the function diagram for h.

(b) Identify the equivalence classes induced by the relation given in Theorem 8.44.

(c) Draw the function diagram for the function $\overline{h}$ as defined in Theorem 8.46.

Problem 8.51. Suppose that $f : \mathbb{R} \to \mathbb{R}$ is a function satisfying $f(x+y) = f(x) + f(y)$ for all $x, y \in \mathbb{R}$.

(a) Prove that $f(0) = 0$.

(b) Prove that $f(-x) = -f(x)$ for all $x \in \mathbb{R}$.

(c) Prove that f is one-to-one if and only if $f^{-1}(\{0\}) = \{0\}$.

(d) Certainly every function given by $f(x) = mx$ for $m \in \mathbb{R}$ satisfies the initial hypothesis. Can you provide an example of a function that satisfies $f(x+y) = f(x) + f(y)$ that is not of the form $f(x) = mx$?

> It is not the critic who counts; not the man who points out how the strong man stumbles, or where the doer of deeds could have done them better. The credit belongs to the man who is actually in the arena, whose face is marred by dust and sweat and blood; who strives valiantly; who errs, who comes short again and again, because there is no effort without error and shortcoming; but who does actually strive to do the deeds; who knows great enthusiasms, the great devotions; who spends himself in a worthy cause; who at the best knows in the end the triumph of high achievement, and who at the worst, if he fails, at least fails while daring greatly, so that his place shall never be with those cold and timid souls who neither know victory nor defeat.
>
> Theodore Roosevelt, statesman & conservationist

8.3 Compositions and Inverse Functions

We begin this section with a method for combining two functions together that have compatible domains and codomains.

Definition 8.52. If $f : X \to Y$ and $g : Y \to Z$ are functions, we define $g \circ f : X \to Z$ via $\boxed{(g \circ f)(x) = g(f(x))}$. The function $g \circ f$ is called the **composition of f and g**.

It is important to notice that the function on the right is the one that "goes first." Moreover, we cannot compose any two random functions since the codomain of the first function must agree with the domain of the second function. In particular, $f \circ g$ may not be a sensible function even when $g \circ f$ exists. Figure 8.4 provides a visual representation of function composition in terms of function diagrams.

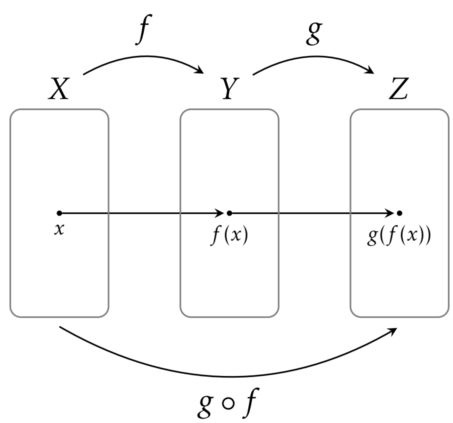


Figure 8.4: Visual representation of function composition.

Problem 8.53. Let $X = \{1,2,3,4\}$ and define $f : X \to X$ and $g : X \to X$ via

$$f = \{(1,1),(2,3),(3,3),(4,4)\}$$

and

$$g = \{(1,1),(2,2),(3,1),(4,1)\}.$$

For each of the following functions, draw the corresponding function diagram in the spirit of Figure 8.4 and identify the range.

(a) $g \circ f$

(b) $f \circ g$

Example 8.54. Consider the inclusion map $\iota : X \to Y$ such that X is a proper subset of Y and suppose $f : Y \to Z$ is a function. Then the composite function $f \circ \iota : X \to Z$ is given by

$$f \circ \iota(x) = f(\iota(x)) = f(x)$$

for all $x \in X$. Notice that $f \circ \iota$ is simply the function f but with a smaller domain. In this case, we say that $f \circ \iota$ is the **restriction of f to X**, which is often denoted by $\boxed{f|_X}$.

The next problem illustrates that $f \circ g$ and $g \circ f$ need not be equal even when both composite functions exist.

Problem 8.55. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$ and $g(x) = 3x-5$, respectively. Determine formulas for the composite functions $f \circ g$ and $g \circ f$.

Problem 8.56. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 5x+7, & \text{if } x < 0 \\ 2x+1, & \text{if } x \geq 0 \end{cases}$$

and $g(x) = 7x - 11$, respectively. Find a formula for the composite function $g \circ f$.

Problem 8.57. Define $f : \mathbb{Z}/15\mathbb{Z} \to \mathbb{Z}/23\mathbb{Z}$ and $g : \mathbb{Z}/23\mathbb{Z} \to \mathbb{Z}/32\mathbb{Z}$ via $f([x]_{15}) = [3x+5]_{23}$ and $g([x]_{23}) = [2x+1]_{32}$, respectively. Find a formula for the composite function $g \circ f$.

The following result provides some insight into where the identity map got its name.

Theorem 8.58. If $f : X \to Y$ is a function, then $f \circ i_X = f = i_Y \circ f$, where i_X and i_Y are the identity maps on X and Y, respectively.

The next theorem tells us that function composition is associative.

Theorem 8.59. If $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Problem 8.60. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing a function diagram is sufficient.

(a) f is surjective, but $g \circ f$ is not surjective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) f is injective, but $g \circ f$ is not injective.

(d) g is injective, but $g \circ f$ is not injective.

Theorem 8.61. If $f : X \to Y$ and $g : Y \to Z$ are both surjective functions, then $g \circ f$ is also surjective.

Theorem 8.62. If $f : X \to Y$ and $g : Y \to Z$ are both injective functions, then $g \circ f$ is also injective.

Corollary 8.63. If $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is also a bijection.

Problem 8.64. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is injective, then g is injective.

(c) If $g \circ f$ is surjective, then f is surjective.

(d) If $g \circ f$ is surjective, then g is surjective.

Theorem 8.65. Let $f : X \to Y$ be a function. Then f is injective if and only if there exists a function $g : Y \to X$ such that $g \circ f = i_X$, where i_X is the identity map on X.

The function g in the previous theorem is often called a **left inverse** of f.

Theorem 8.66. Let $f : X \to Y$ be a function. Then f is surjective if and only if there exists a function $g : Y \to X$ such that $f \circ g = i_Y$, where i_Y is the identity map on Y.

The function g in the previous theorem is often called a **right inverse** of f.

Problem 8.67. Provide an example of a function that has a left inverse but does not have a right inverse. Find the left inverse of your proposed function.

Problem 8.68. Provide an example of a function that has a right inverse but does not have a left inverse. Find the right inverse of your proposed function.

Corollary 8.69. If $f : X \to Y$ and $g : Y \to X$ are functions satisfying $g \circ f = i_X$ and $f \circ g = i_Y$, then f is a bijection.

In the previous result, the functions f and g "cancel" each other out. In this case, we say that g is a **two-sided inverse** of f.

Definition 8.70. Let $f : X \to Y$ be a function. The relation f^{-1} from Y to X, called f **inverse**, is defined via

$$\boxed{f^{-1} = \{(f(x), x) \in Y \times X \mid x \in X\}}.$$

Notice that we called f^{-1} a relation and not a function. In some circumstances f^{-1} will be a function and sometimes it will not be. Given a function f, the inverse relation is simply the set of ordered pairs that results from reversing the ordered pairs in f. It is worth pointing out that we have only defined inverse relations for functions. However, one can easily adapt our definition to handle arbitrary relations.

Problem 8.71. Consider the function f given in Example 8.2 (see Figure 8.1). List the ordered pairs in the relation f^{-1} and draw the corresponding digraph. Is f^{-1} a function?

Problem 8.72. Provide an example of a function $f : X \to Y$ such that f^{-1} is a function. Drawing a function diagram is sufficient.

Problem 8.73. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. What is the relationship between the graph of the function f and the graph of the inverse relation f^{-1}?

Theorem 8.74. Let $f : X \to Y$ be a function. Then $f^{-1} : Y \to X$ is a function if and only if f is a bijection.

Problem 8.75. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate phrase.

> The relation f^{-1} is a function if and only if every horizontal line hits the graph of f ________________.

Explain why this statement is true.

Theorem 8.76. If $f : X \to Y$ is a bijection, then

(a) $f^{-1} \circ f = i_X$, and

(b) $f \circ f^{-1} = i_Y$.

Theorem 8.77. If $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ is also a bijection.

Theorem 8.78. If $f : X \to Y$ and $g : Y \to X$ are functions such that $g \circ f = i_X$ and $f \circ g = i_Y$, then f^{-1} is a function and $g = f^{-1}$.

The upshot of Theorems 8.76 and 8.78 is that if f^{-1} is a function, then it is the only one satisfying the two-sided inverse property exhibited in Corollary 8.69 and Theorem 8.76. That is, inverse functions are unique when they exist. When the relation f^{-1} is a function, we call it the **inverse function** of f.

Theorem 8.79. If $f : X \to Y$ is a bijection, then $(f^{-1})^{-1} = f$.

In the previous theorem, we restricted our attention to bijections so that f^{-1} would be a function, thus making $(f^{-1})^{-1}$ a sensible inverse relation in light of Definition 8.70. If we had defined inverses for arbitrary relations, then we would not have needed to require the function in Theorem 8.79 to be a bijection. In fact, we do not even need to require the relation to be a function. That is, if R is a relation from X to Y, then $(R^{-1})^{-1} = R$, as expected. Similarly, the next result generalizes to arbitrary relations.

Theorem 8.80. If $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

The previous theorem is sometimes referred to as the "socks and shoes theorem". Do you see how it got this name?

> The most difficult thing is the decision to act.
> The rest is merely tenacity.
>
> Amelia Earhart, aviation pioneer

8.4 Images and Preimages of Functions

There are two important types of sets related to functions.

Definition 8.81. Let $f : X \to Y$ be a function.

(a) If $S \subseteq X$, the **image** of S under f is defined via

$$\boxed{f(S) := \{f(x) \mid x \in S\}}.$$

(b) If $T \subseteq Y$, the **preimage** (or **inverse image**) of T under f is defined via

$$\boxed{f^{-1}(T) := \{x \in X \mid f(x) \in T\}}.$$

The image of a subset S of the domain is simply the subset of the codomain we obtain by mapping the elements of S. It is important to emphasize that the function f maps *elements* of X to *elements* of Y, but we can apply f to a *subset* of X to yield a *subset* of Y. That is, if $S \subseteq X$, then $f(S) \subseteq Y$. Note that the image of the domain is the same as the range of the function. That is, $f(X) = \mathrm{Rng}(f)$.

When it comes to preimages, there is a real opportunity for confusion. In Section 8.3, we introduced the inverse relation f^{-1} of a function f (see Defintion 8.70) and proved that this relation is a function exactly when f is a bijection (see Theorem 8.74). If $f^{-1} : Y \to X$ is a function, then it is sensible to write $f^{-1}(y)$ for $y \in Y$. Notice that we defined the preimage of a subset of the codomain regardless of whether f^{-1} is a function or not. In particular, for $T \subseteq Y$, $f^{-1}(T)$ is the set of elements in the domain that map to elements in T. As a special case, $f^{-1}(\{y\})$ is the set of elements in the domain that map to $y \in Y$. If $y \notin \mathrm{Rng}(f)$, then $f^{-1}(\{y\}) = \emptyset$. Notice that if $y \in Y$, $f^{-1}(\{y\})$ is always a sensible thing to write while $f^{-1}(y)$ only makes sense if f^{-1} is a function. Also, note that the preimage of the codomain is the domain. That is, $f^{-1}(Y) = X$.

Problem 8.82. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. List elements in each of the following sets.

(a) $f(\{0, 1, 2\})$

(b) $f^{-1}(\{0, 1, 4\})$

Problem 8.83. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 3x^2 - 4$. Find each of the following sets.

(a) $f(\{-1, 1\})$

(b) $f([-2, 4])$

(c) $f((-2, 4))$

(d) $f^{-1}([-10, 1])$

(e) $f^{-1}((-3, 3))$

(f) $f(\emptyset)$

(g) $f(\mathbb{R})$

(h) $f^{-1}(\{-1\})$

(i) $f^{-1}(\emptyset)$

(j) $f^{-1}(\mathbb{R})$

Problem 8.84. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$.

(a) Find two nonempty subsets A and B of $\mathbb{R}$ such that $A \cap B = \emptyset$ but $f^{-1}(A) = f^{-1}(B)$.

(b) Find two nonempty subsets A and B of $\mathbb{R}$ such that $A \cap B = \emptyset$ but $f(A) = f(B)$.

Problem 8.85. Consider the equivalence relation given in Theorem 8.44. Explain why each equivalence class $[a]$ is equal to $f^{-1}(\{f(a)\})$.

Problem 8.86. Suppose $f : X \to Y$ is an injection and A and B are disjoint subsets of X. Are $f(A)$ and $f(B)$ necessarily disjoint subsets of Y? If so, prove it. Otherwise, provide a counterexample.

Problem 8.87. Find examples of functions f and g together with sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$.

Problem 8.88. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$ and $C, D \subseteq Y$. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $f(f^{-1}(C)) \subseteq C$.

(n) $f(f^{-1}(C)) \supseteq C$.

Problem 8.89. For each of the statements in the previous problem that were false, determine conditions, if any, on the corresponding sets that would make the statement true.

We can generalize the results above to handle arbitrary collections of sets.

Theorem 8.90. Let $f : X \to Y$ be a function and suppose $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of subsets of X.

(a) $f\left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} f(A_\alpha)$.

(b) $f\left(\bigcap_{\alpha\in\Delta} A_\alpha\right) \subseteq \bigcap_{\alpha\in\Delta} f(A_\alpha)$.

Theorem 8.91. Let $f : X \to Y$ be a function and suppose $\{C_\alpha\}_{\alpha\in\Delta}$ is a collection of subsets of Y.

(a) $f^{-1}\left(\bigcup_{\alpha\in\Delta} C_\alpha\right) = \bigcup_{\alpha\in\Delta} f^{-1}(C_\alpha)$.

(b) $f^{-1}\left(\bigcap_{\alpha\in\Delta} C_\alpha\right) = \bigcap_{\alpha\in\Delta} f^{-1}(C_\alpha)$.

> The obstacle is the path.
>
> Zen saying, Author Unknown

8.5 Continuous Real Functions

In this section, we will explore the concept of continuity, which you likely encountered in high school.

Definition 8.92. A **real function** is any function $f : A \to \mathbb{R}$ such that A is a nonempty subset of $\mathbb{R}$.

There are several equivalent definitions of continuity for real functions. The following characterization is typically referred to as the **epsilon-delta definition of continuity**. Our definition mimics the definition of continuity used in metric spaces, which $\mathbb{R}$ equipped with absolute value happens to be an example of. Recall that $|a-b| < r$ means that the distance between a and b is less than r (see discussion below Corollary 5.31).

Definition 8.93. Suppose f is a real function such that $a \in \mathrm{Dom}(f)$. We say that f is **continuous at** a if for every $\varepsilon > 0$, there exists $\delta > 0$ such that if $x \in \mathrm{Dom}(f)$ and $|x-a| < \delta$, then $|f(x) - f(a)| < \varepsilon$. If f is continuous at every point in $B \subseteq \mathrm{Dom}(f)$, then we say that f is **continuous on** B. If f is continuous on its entire domain, we simply say that f is **continuous**.

Loosely speaking, a real function f is continuous at the point $a \in \mathrm{Dom}(f)$ if we can get $f(x)$ arbitrarily close to $f(a)$ by considering all $x \in \mathrm{Dom}(f)$ sufficiently close to a. The value ε is indicating how close to $f(a)$ we need to be while the value δ is providing the "window" around a needed to guarantee that all points in the window (and in the domain) yield outputs within ε of $f(a)$. Figure 8.5 illustrates our definition of continuity. Note that in the figure, the point a is fixed while we need to consider all $x \in \mathrm{Dom}(f)$ such that $|x-a| < \delta$. The dashed box in the figure has dimensions 2δ by 2ε and is centered at the point $(a, f(a))$. Intuitively, the function is continuous at a since given $\varepsilon > 0$, we could find $\delta > 0$ so that the graph of the function never exits the top or bottom of the dashed box.

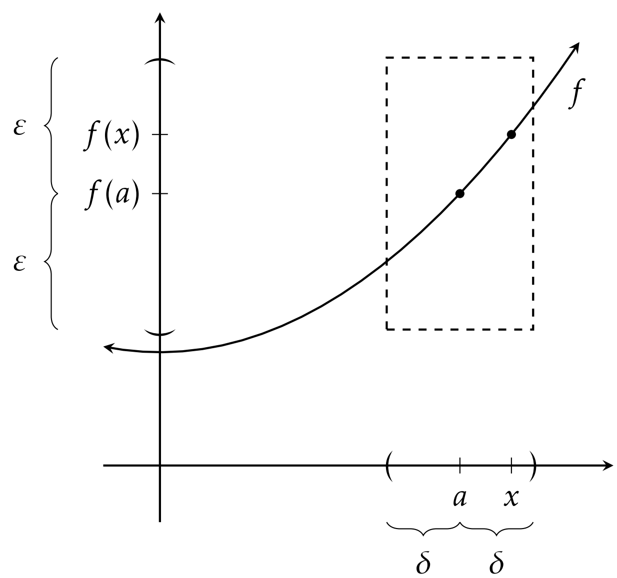


Figure 8.5: Visual representation of continuity of f at a.

Perhaps you have encountered the phrase "a function is continuous if you can draw its graph without lifting your pencil." While this description provides some intuition about what continuity of a function means, it is neither accurate nor precise enough to capture the meaning of continuity.

When proving that a function is continuous at a point, the choice of δ depends on both the point in question and the value of ε. An example should be helpful.

Example 8.94. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 3x + 2$. Let's prove that f is continuous (at every point in the domain). Let $a \in \mathbb{R}$ and let $\varepsilon > 0$. Choose $\delta = \varepsilon/3$. We will see in a moment why this is a good choice for δ. Suppose $x \in \mathbb{R}$ such that $|x-a| < \delta$. We see that

$$|f(x) - f(a)| = |(3x+2) - (3a+2)| = |3x - 3a| = 3 \cdot |x-a| < 3 \cdot \delta = 3 \cdot \varepsilon/3 = \varepsilon.$$

We have shown that f is continuous at a, and since a was arbitrary, f is continuous.

Problem 8.95. Prove that each of the following real functions is continuous using Definition 8.93.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined via $g(x) = x + 42$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = 5x$.

The next result tells us that every linear real function is continuous. Do not forget to handle the case when $m = 0$ in your proof. Note that the case when $m = 0$ proves that every constant function is continuous.

Theorem 8.96. If $f : \mathbb{R} \to \mathbb{R}$ is defined via $f(x) = mx + b$ for $m, b \in \mathbb{R}$, then f is continuous.

The second part of the next problem is much harder than you might expect.

Problem 8.97. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$.

(a) Prove that f is continuous at 0.

(b) Prove that f is continuous at 1.

Problem 8.98. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = \sqrt{x}$. Prove that f is continuous at 0.

Problem 8.99. Suppose f is a real function. Write a precise statement for what it means for f to not be continuous at $a \in \mathrm{Dom}(f)$.

Problem 8.100. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x = 0 \\ x, & \text{otherwise.} \end{cases}$$

Determine where f is continuous and justify your assertion.

Problem 8.101. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Determine where f is continuous and justify your assertion.

After completing the next problem, reflect on the statement "a function is continuous if you can draw its graph without lifting your pencil."

Problem 8.102. Define $f : \mathbb{N} \to \mathbb{R}$ via $f(x) = 1$. Notice the domain! Determine where f is continuous and justify your assertion.

Theorem 8.103. Suppose f is a real function. Then f is continuous if and only if the preimage $f^{-1}(U)$ of every open set U is an open set intersected with the domain of f.

The previous characterization of continuity is often referred to as the "open set definition of continuity," although for us it is a theorem instead of a definition. This is the definition used in topology. Another notion of continuity, called "sequential continuity", makes use of convergent sequences. All of these characterizations of continuity are equivalent for the real numbers (using the standard definition of an open set). However, there are contexts in mathematics where the epsilon-delta definition of continuity is undefined (because there is not a notion of distance in either the domain or codomain) and others where continuity and sequential continuity are not equivalent.

Since every open set is the union of bounded open intervals (Definition 5.53), the union of open sets is open (Theorem 5.58), and preimages respect unions (Theorem 8.91), we can strengthen Theorem 8.103 into a slightly more useful result.

Theorem 8.104. Suppose f is a real function. Then f is continuous if and only if the preimage $f^{-1}(I)$ of every bounded open interval I is an open set intersected with the domain of f.

Now that we have two methods for verifying continuity (Definition 8.93 and Theorem 8.103/8.104), you can use either one when approaching the remaining problems in this section. Sometimes it does not matter which approach you take and other times one method might be better suited to the task.

Problem 8.105. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is continuous.

Problem 8.106. Define $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ via $f(x) = \frac{1}{x}$. Determine where f is continuous and justify your assertion.

The previous problems once again calls into question the phrase "a function is continuous if you can draw its graph without lifting your pencil."

Problem 8.107. Find a continuous real function f and an open interval I such that the preimage $f^{-1}(I)$ is not an open interval.

For the next few problems, if you attempt to construct counterexamples, you may rely on your previous knowledge about various functions that you encountered in high school and calculus.

Problem 8.108. Suppose f is a continuous real function. If U is an open set contained in $\text{Dom}(f)$, is the image $f(U)$ always open? If so, prove it. Otherwise, provide a counterexample.

Problem 8.109. Suppose f is a continuous real function. If C is a closed set, is the preimage $f^{-1}(C)$ always a closed set? If so, prove it. Otherwise, provide a counterexample.

Problem 8.110. Suppose f is a continuous real function. If $[a,b]$ is a closed interval contained in $\text{Dom}(f)$, is the image $f([a,b])$ always a closed interval? If so, prove it. Otherwise, provide a counterexample.

Problem 8.111. Suppose f is a continuous real function. If C is a closed set contained in $\mathrm{Dom}(f)$, is the image $f(C)$ always a closed set? If so, prove it. Otherwise, provide a counterexample.

Problem 8.112. Suppose f is a continuous real function. If B is bounded set contained in $\mathrm{Dom}(f)$, is the image $f(B)$ always a bounded set? If so, prove it. Otherwise, provide a counterexample.

Problem 8.113. Suppose f is a continuous real function. If B is a bounded set, is the preimage $f^{-1}(B)$ always a bounded set? If so, prove it. Otherwise, provide a counterexample.

Problem 8.114. Suppose f is a continuous real function. If K is a compact set, is the preimage $f^{-1}(B)$ always a compact set? If so, prove it. Otherwise, provide a counterexample.

Problem 8.115. Suppose f is a continuous real function. If C is a connected set contained in $\mathrm{Dom}(f)$, is the image $f(C)$ always connected? If so, prove it. Otherwise, provide a counterexample.

Problem 8.116. Suppose f is a continuous real function. If C is a connected set, is the preimage $f^{-1}(C)$ always a connected set? If so, prove it. Otherwise, provide a counterexample.

Perhaps you noticed the absence of one natural question in the previous sequence of problems. If f is a continuous real function and K is a subset of the domain of f, is the image $f(K)$ a compact set? It turns out that the answer is "yes", but proving this fact is beyond the scope of this book. This theorem is often proved in a real analysis course and is then used to prove the Extreme Value Theorem, which you may have encountered in your calculus course.

The next result is a special case of the well-known **Intermediate Value Theorem**, which states that if f is a continuous real function whose domain contains the interval $[a,b]$, then f attains every value between $f(a)$ and $f(b)$ at some point within the interval $[a,b]$. To prove the special case, utilize Theorem 5.87 and Problem 8.115 together with a proof by contradiction.

Theorem 8.117. Suppose f is a real function. If f is continuous on $[a,b]$ such that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists $r \in [a,b]$ such that $f(r) = 0$.

If we generalize the previous result, we obtain the Intermediate Value Theorem.

Theorem 8.118 (Intermediate Value Theorem)**.** Suppose f is a real function. If f is continuous on $[a,b]$ such that $f(a) < c < f(b)$ or $f(a) > c > f(b)$ for some $c \in \mathbb{R}$, then there exists $r \in [a,b]$ such that $f(r) = c$.

Problem 8.119. Is the converse of the Intermediate Value Theorem true? If so, prove it. Otherwise, provide a counterexample.

> The miracle of the appropriateness of the language of mathematics for the formulation of the laws of physics is a wonderful gift which we neither understand nor deserve. We should be grateful for it and hope that it will remain valid in future research and that it will extend, for better or for worse, to our pleasure, even though perhaps also to our bafflement, to wide branches of learning.
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> Eugene Paul Wigner, theoretical physicist