Chapter 7

Functions

7.1 Introduction to Functions

Undoubtably, you have encountered the concept of function in your prior mathematical experience. In this section, we will introduce the concept of function as a special type of relation. As you shall see, this agrees with any previous definition of function that you may have learned.

Up until this point, you've probably only encountered functions as an algebraic rule, e.g., $f(x) = x^2 - 1$, for transforming one real number into another. However, we can study functions in a much broader context. Loosely speaking, the basic building blocks of a function are a first set and a second sets, say X and Y, respectively, and a "correspondence" that assigns each element of X to exactly one element of Y. Let's take a look at the actual definition.

Definition 7.1. Let X and Y be two nonempty sets. A **function** from set X to set Y, denoted $f : X \to Y$, is a relation (i.e., subset of $X \times Y$) such that:

(a) For each $x \in X$, there exists $y \in Y$ such that $(x, y) \in f$, and

(b) If $(x, y_1), (x, y_2) \in f$, then $y_1 = y_2$.

Note that if $(x, y) \in f$, we usually write $y = f(x)$ and say that "f maps x to y."

Part (a) of Definition 7.1 says that every element of X appears in the first coordinate of an ordered pair in the relation. Part (b) says that each element of X only appears once in the first coordinate of an ordered pair in the relation. It is important to note that there are no restrictions on whether an element of Y ever appears in the second coordinate. Furthermore, if an element of Y appears in the second coordinate, it may appear again in a different ordered pair.

Definition 7.2. The set X from Definition 7.1 is called the **domain** of f and is denoted by $\mathrm{Dom}(f)$. The set Y is called the **codomain** of f and is denoted by $\mathrm{Codom}(f)$. The set

$$\mathrm{Rng}(f) = \{y \in Y \mid \text{there exists } x \text{ such that } y = f(x)\}$$

is called the **range** of f or the **image of** X under f. If f is a function and $(x, y) \in f$, then we may refer to x as the **input** of f and y as the **output** of f.

It follows immediately from the definition that $\mathrm{Rng}(f) \subseteq \mathrm{Codom}(f)$. However, it is possible that the range of f is a proper subset of the codomain.

Exercise 7.3. Let $X = \{\circ, \Box, \triangle, ☺\}$ and $Y = \{a, b, c, d, e\}$. Determine whether each of the following represent functions. Explain. If the relation is a function, determine the domain, codomain, and range.

(a) $f : X \to Y$ defined via $f = \{(\circ, a), (\Box, b), (\triangle, c), (☺, d)\}$.

(b) $g : X \to Y$ defined via $g = \{(\circ, a), (\Box, b), (\triangle, c), (☺, c)\}$.

(c) $h : X \to Y$ defined via $h = \{(\circ, a), (\Box, b), (\triangle, c), (\circ, d)\}$.

(d) $k : X \to Y$ defined via $k = \{(\circ, a), (\Box, b), (\triangle, c), (☺, d), (\Box, e)\}$.

(e) $l : X \to Y$ defined via $l = \{(\circ, e), (\Box, e), (\triangle, e), (☺, e)\}$.

(f) $m : X \to Y$ defined via $m = \{(\circ, a), (\triangle, b), (☺, c)\}$.

(g) $\mathrm{happy} : Y \to X$ defined via $\mathrm{happy}(y) = ☺$ for all $y \in Y$.

(h) $\mathrm{id} : X \to X$ defined via $\mathrm{id}(x) = x$ for all $x \in X$.

(i) $\mathrm{nugget} : X \to X$ defined via

$$\mathrm{nugget}(x) = \begin{cases} x, & \text{if } x \text{ is a geometric shape,} \\ \Box, & \text{otherwise.} \end{cases}$$

One useful representation of functions on finite sets is via **bubble diagrams**. To draw a bubble diagram for a function $f : X \to Y$, draw one circle (i.e, a "bubble") for each of X and Y and for each element of each set, put a dot in the corresponding set. Typically, we draw X on the left and Y on the right. Next, draw an arrow from $x \in X$ to $y \in Y$ if $f(x) = y$ (i.e., $(x, y) \in f$). Note that we can draw bubble diagrams even if f is not a function.

Example 7.4. Figure 7.1 depicts a bubble diagram for a function from domain $X = \{a, b, c, d\}$ to codomain $Y = \{1, 2, 3, 4\}$. In this case, the range is equal to $\{1, 2, 4\}$.

Exercise 7.5. For each of the relations in Exercise 7.3 draw the corresponding bubble diagram.

Problem 7.6. What properties does a bubble diagram have to have in order to represent a function?

Exercise 7.7. Provide an example of each of the following. You may draw a bubble diagram, write down a list of ordered pairs, or a write a formula (as long as the domain and codomain are clear).

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

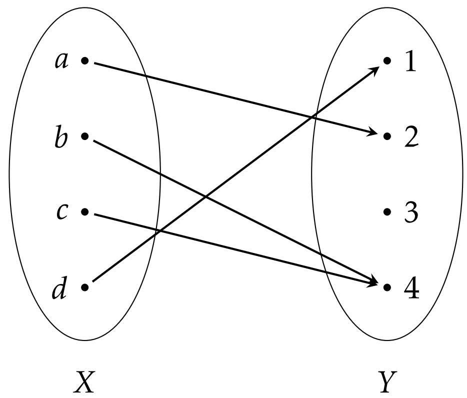

Figure 7.1: An example of a bubble diagram for a function.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\text{Rng}(g)$ is strictly smaller than $\text{Codom}(g)$.

Problem 7.8. Let $f : X \to Y$ be a function and suppose that X and Y are finite sets with n and m elements, respectively, such that $n < m$. Is it possible for $\text{Rng}(f) = \text{Codom}(f)$? Explain.

Problem 7.9. In high school I am sure that you were told that a graph represents a function if it passes the **vertical line test**. Using our terminology of ordered pairs, explain why this works.

Definition 7.10. Two functions are equal if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, if $f : X \to Y$ and $g : X \to Y$ are functions, then $f = g$ if and only if $f(x) = g(x)$ for all $x \in X$.

If two functions are defined by the same algebraic formula, but have different domains, then they are *not* equal. For example, the function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2$ is not equal to the function $g : \mathbb{N} \to \mathbb{N}$ defined via $g(x) = x^2$.

Definition 7.11. Let $f : X \to Y$ be a function.

(a) The function f is said to be **one-to-one** (or **injective**) if for all $y \in \text{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.

(b) The function f is said to be **onto** (or **surjective**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.

(c) If f is both one-to-one and onto, we say that f is a **bijection** (or **one-to-one correspondence**).

Remark 7.12. Let $f : X \to Y$ be a function. To prove that f is one-to-one, start by assuming that $f(x_1) = f(x_2)$ and then work to show that $x_1 = x_2$. That is, a function f is one-to-one if and only if for all $x_1, x_2 \in X$, if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show that f is onto, you should start with an arbitrary $y \in Y$ and then work to show that there exists $x \in X$ such that $y = f(x)$.

Exercise 7.13. Provide an example of each of the following. You may draw a bubble diagram, write down a list of ordered pairs, or write a formula (as long as the domain and codomain are clear). Assume that X and Y are finite sets.

(a) A function $f : X \to Y$ that is one-to-one but not onto.

(b) A function $f : X \to Y$ that is onto but not one-to-one.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither one-to-one nor onto.

Problem 7.14. Perhaps you've heard of the **horizontal line test** (i.e., every horizontal line hits the graph of $f : \mathbb{R} \to \mathbb{R}$ at most once). What is the horizontal line test testing for?

Exercise 7.15. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is one-to-one but not onto.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is onto but not one-to-one.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is a bijection.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither one-to-one nor onto.

Exercise 7.16. Determine which of the following functions are one-to-one, onto, both, or neither. In each case, you should provide proofs and counterexamples as appropriate.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2$

(b) $g : \mathbb{R} \to [0, \infty)$ defined via $g(x) = x^2$

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = x^3$

(d) $k : \mathbb{R} \to \mathbb{R}$ defined via $k(x) = x^3 - x$

(e) $l : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ defined via $l(x_1, x_2) = x_1^2 + x_2^2$

(f) $N : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ defined via $N(n) = (n, n)$

Definition 7.17. If X is a nonempty set, then the function $i_X : X \to X$ defined via $i_X(x) = x$ is called the *identity function* on X.

Theorem 7.18. The identity function on a nonempty set X is a bijection.

Exercise 7.19. Let A and B be sets and let $S \subseteq A \times B$. Define $\pi_1 : S \to A$ and $\pi_2 : S \to B$ via $\pi_1(a, b) = a$ and $\pi_2(a, b) = b$. We call π_1 (respectively, π_2) the **projections** of S onto A (respectively, B).

(a) Provide examples to show that π_1 does not need to be one-to-one or onto.

(b) Suppose that S is a function (recall that a function is a set of ordered pairs, so this makes sense). Is π_1 one-to-one? Is π_1 onto? How about π_2?

Theorem 7.20. Let A be a nonempty set and suppose $\sim$ is an equivalence relation on A. Then the function $\phi : A \to A/{\sim}$ defined via $\phi(x) = [x]$ is onto.[1]

7.2 Images and Inverse Images of Functions

There are two important sets related to functions.

Definition 7.21. Let $f : X \to Y$ be a function.

(a) If $S \subseteq X$, the **image** of S under f is defined via

$$f(S) := \{f(x) \mid x \in S\}.$$

(b) If $T \subseteq Y$, the **inverse image** (or **preimage**) of T under f is defined via

$$f^{-1}(T) := \{x \in X \mid f(x) \in T\}.$$

You've likely encountered inverse *functions* before. But in this context, we are discussing inverse *images*. It's important to point out that the use of the notation f^{-1} does not make any assumptions about whether the inverse function exists. We will tackle inversion functions in the next section.

Note that the image of the domain is the same as its range. That is, $f(X) = \mathrm{Rng}(f)$. Moreover, the inverse image of the codomain is the domain. That is, $f^{-1}(Y) = X$.

Exercise 7.22. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. Find $f(\{-2,-1,0,1,2\})$ and $f^{-1}(\{0,1,4\})$.

Exercise 7.23. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = 3x^2 - 4$. Find each of the following.

(a) $f([-2,4])$

(b) $f((-2,4))$

(c) $f^{-1}([-10,1])$

(d) $f^{-1}((-3,3))$

(e) $f(\emptyset)$

(f) $f(\mathbb{R})$

(g) $f^{-1}(\emptyset)$

(h) $f^{-1}(\mathbb{R})$

[1] Recall that $A/{\sim}$ is the set of equivalence classes induced by the equivalence relation $\sim$.

(i) Find two non-empty subsets A, B of $\mathbb{R}$ such that $A \cap B = \emptyset$ but $f^{-1}(A) = f^{-1}(B)$

(j) Find two non-empty subsets A, B of $\mathbb{R}$ such that $A \cap B = \emptyset$ but $f(A) = f(B)$

Problem 7.24. Find examples of functions f and g together with sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$.

Problem 7.25. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$ and $C, D \subseteq Y$. Determine whether each of the following statements is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $f(f^{-1}(C)) \subseteq C$.

(n) $f(f^{-1}(C)) \supseteq C$.

Exercise 7.26. For each of the statements in previous problem that were false, determine conditions—if any—on the corresponding sets that would make the statement true.

7.3 Compositions and Inverse Functions

Definition 7.27. If $f : X \to Y$ and $g : Y \to Z$ are functions, then a new function $g \circ f : X \to Z$ can be defined by $(g \circ f)(x) = g(f(x))$ for all $x \in \mathrm{Dom}(f)$.

It is important to notice that the function on the right is the one that "goes first."

Exercise 7.28. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing bubble diagrams is sufficient.

(a) f is onto, but $g \circ f$ is not onto.

(b) g is onto, but $g \circ f$ is not onto.

(c) f is one-to-one, but $g \circ f$ is not one-to-one.

(d) g is one-to-one, but $g \circ f$ is not.

Theorem 7.29. If $f : X \to Y$ and $g : Y \to Z$ are both functions that are onto, then $g \circ f$ is also onto.

Theorem 7.30. If $f : X \to Y$ and $g : Y \to Z$ are both functions that are one-to-one, then $g \circ f$ is also one-to-one.

Corollary 7.31. If $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is also a bijection.

Problem 7.32. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. Determine whether each of the following statements is true or false. If the statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is one-to-one, then f is one-to-one.

(b) If $g \circ f$ is one-to-one, then g is one-to-one.

(c) If $g \circ f$ is onto, then f is onto.

(d) If $g \circ f$ is onto, then g is onto.

The next theorem tells us that function composition is associative.

Theorem 7.33. If $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Theorem 7.34. Let $f : X \to Y$ be a function. Then f is one-to-one if and only if there exists a function $g : Y \to X$ such that $g \circ f = i_X$, where i_X is the identity function on X.

The function g in the previous theorem is often called a *left inverse* of f.

Theorem 7.35. Let $f : X \to Y$ be a function. Then f is onto if and only if there exists a function $g : Y \to X$ such that $f \circ g = i_Y$, where i_Y is the identity function on Y.

The function g in the previous theorem is often called a *right inverse* of f.

Exercise 7.36. Provide an example of a function that has a left inverse but does not have a right inverse. Find the left inverse of your proposed function.

Exercise 7.37. Provide an example of a function that has a right inverse but does not have a left inverse. Find the right inverse of your proposed function.

Corollary 7.38. If $f : X \to Y$ and $g : Y \to X$ are functions satisfying $g \circ f = i_X$ and $f \circ g = i_Y$, then f is a bijection.

In the previous result, the functions f and g "cancel" each other out. We say that g is a *two-sided inverse* of f.

Definition 7.39. Let $f : X \to Y$ be a function. The relation f^{-1}, called f **inverse**, is defined via

$$f^{-1} = \{(f(x), x) \in Y \times X \mid x \in X\}.$$

Notice that we called f^{-1} a relation and not a function. In some circumstances f^{-1} will be a function and sometimes it will not be.

Exercise 7.40. Provide an example of a function $f : X \to Y$ such that f^{-1} is *not* a function. A bubble diagram is sufficient.

Exercise 7.41. Provide an example of a function $f : X \to Y$ such that f^{-1} is a function. A bubble diagram is sufficient.

Theorem 7.42. Let $f : X \to Y$ be a function. Then f^{-1} is a function if and only if f is a bijection.

Theorem 7.43. If $f : X \to Y$ is a bijection, then

(a) $f^{-1} \circ f = i_X$, and

(b) $f \circ f^{-1} = i_Y$.

Theorem 7.44. Let $f : X \to Y$ and $g : Y \to X$ be functions such that f is a bijection. If $g \circ f = i_X$ and $f \circ g = i_Y$, then $g = f^{-1}$.

The upshot of the previous two theorems is that if f^{-1} is a function, then it is the only one satisfying the two-sided inverse property exhibited in Corollary 7.38 and Theorem 7.43.

Theorem 7.45. If $f : X \to Y$ is a bijection, then $f^{-1} : Y \to X$ is a bijection and $(f^{-1})^{-1} = f$.

Theorem 7.46. If $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $(g \circ f)^{-1} = f^{-1} \circ g^{-1}$.

The previous theorem is sometimes referred to as the "socks and shoes theorem". Do you see how it got this name?

Theorem 7.47. Let $f : X \to Y$ be a function and define $\sim$ on X via $a \sim b$ if and only if $f(a) = f(b)$.

(a) The relation $\sim$ is an equivalence relation,

(b) Each equivalence class $[a]$ is equal to $f^{-1}(f(a))$,

(c) The function $g : X/\sim \to f(X)$ defined via $g([a]) = f(a)$ is a bijection.