Exam 1 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 1 is worth a total of 28 points and is due at the beginning of class on **Wednesday, October 4**. Your total combined score on the in-class portion and take-home portion is worth 18% of your overall grade.

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

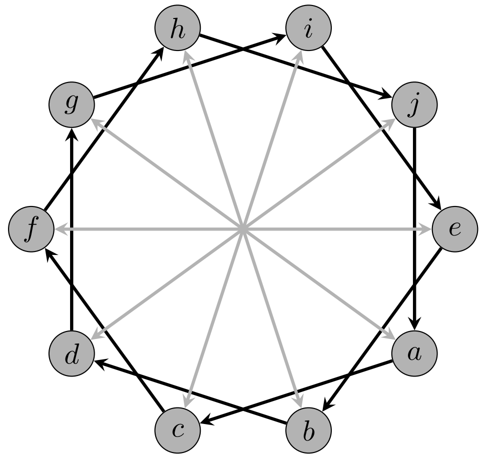
You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. (2 points) Provide an example of a group $(G, *)$ and elements $a, b \in G$ such that $(a * b)^2 \neq a^2 * b^2$.
2. (2 points) Suppose G is the group given by the Cayley diagram below and let R_{10} denote the group of rotational symmetries of a regular decagon (10 sides). You may assume that e is the identity. Determine whether G and R_{10} are isomorphic and justify your answer.



3. (4 points each) Prove **two** of the following theorems. For each theorem, assume that $(G, *)$ is a group and e is the identity element.

   Theorem 1. Assume $(G, \star)$ is a group and let H be a nonempty subset of G that is (i) closed under $\star$ and (ii) closed under inverses (i.e., for all $h, k \in H$, (i) $hk \in H$ and (ii) $h^{-1} \in H$). Then $H \leq G$.*

   Theorem 2. Let $x \in G$. Then $x^m = e$ iff $|\langle x \rangle|$ divides m.

   Theorem 3. If $x \in G \setminus \{e\}$ such that $x^n \neq e$ for all $n \in \mathbb{Z}^+$, then $x^i \neq x^j$ for all $i \neq j$.

4. Suppose $(G, *)$ is a group and let $H, K \leq G$.
   (a) (4 points) Prove that $H \cap K \leq G$.†
   (b) (2 points) Can we replace intersection with union in the theorem above? If so, prove the corresponding theorem. If not, then provide a specific counterexample.‡
5. (2 points each) Suppose $(G, *)$ and $(H, \circ)$ are groups. Define $\star$ on $G \times H$ via $(g_1, h_1) \star (g_2, h_2) = (g_1 * g_2, h_1 \circ h_2)$.§ Suppose e_G and e_H are the identity elements of G and H, respectively. It turns out that $(G \times H, \star)$ is a group. If you need to touch up on your knowledge of Cartesian products of sets, see Appendix A of the course notes.
   (a) Prove that $G \times H$ is closed under $\star$.
   (b) What is the identity element of $G \times H$? Verify that this element is in fact the identity.
   (c) Let $(g, h) \in G \times H$. What is $(g, h)^{-1}$? Verify that this element is in fact the inverse of (g, h).
   (d) Consider $S_2 \times R_4$ (using the operation of S_2 in the first component and the operation of R_4 in the second component). Find a generating set for $S_2 \times R_4$ and then create a Cayley diagram for this group.
   (e) Argue that $S_2 \times R_4$ cannot be isomorphic to any of D_4, R_8, Q_8, and L_3.¶

*This is Theorem 5.53.
†This is Theorem 5.59.
‡This is Problem 5.60.
§This looks fancier than it is. We're just doing the operation of each group in the appropriate component.
¶The group L_3 is the group that acts on 3 light switches. The upshot of this last problem is that there are at least 5 distinct groups of order 8 up to isomorphism. In turns out that there aren't any others (up to isomorphism).

 Written by D.C. Ernst