Exam 3 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 3 is worth a total of 19 points and is worth 30% of your overall score on Exam 3. This take-home exam is due at the beginning of class on **Wednesday, November 21**. Your overall score on Exam 3 is worth 18% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 5.35, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

1. (4 points) On the in-class portion of Exam 3, you were asked to prove one of the following theorems. Prove **one** of the theorems you did not attempt on the in-class exam. You may use an earlier theorem to prove a later theorem.

   (a) Assume G is a finite abelian group and $x, y \in G$ such that $|x| = m$ and $|y| = n$. If $\gcd(m, n) = 1$, then $|xy| = mn$.*

   (b) Assume G is a finite abelian group. If n is the maximal order among the elements in G, then the order of every element in G divides n.†

   (c) If G is a finite abelian group with at most one subgroup of any size, then G is cyclic.‡

For the remaining problems, you can use any theorems in Chapters 5 and 6 as long as they come before the problems I am asking you to complete.

2. (3 points) Complete **one** of the following.

   (a) Problem 5.33

   (b) Problem 6.28

3. (4 points) Prove **one** of the following.

   (a) Theorem 5.35

   (b) Theorem 6.39

4. (4 points) Let G be a group. Since the elements of the center $Z(G)$ commute with all the elements of G, the left and right cosets of $Z(G)$ will be equal, and hence $Z(G)$ is normal in G. Prove that if $G/Z(G)$ is cyclic, then G is abelian.

5. (4 points) The following definition will be useful for two of the problems below. Suppose $\phi : G_1 \to G_2$ is a function between two groups that satisfies the homomorphic property (but may or may not be one-to-one or onto). Define the **kernel** of ϕ via

$$\ker(\phi) := \{g \in G_1 \mid \phi(g) = e_2\},$$

   where e_1 and e_2 are the identities of G_1 and G_2, respectively. Note that we denoted the kernel by K_ϕ on the in-class portion of Exam 2. By the in-class portion of Exam 2, $\ker(\phi)$ is a subgroup of G_1. Complete **one** of the following.

   (a) Prove that $\ker(\phi)$ is a normal subgroup of G_1.§

   (b) Prove that if $\ker(\phi) = \{e_1\}$, then the function ϕ given above is one-to-one.¶

   (c) Prove that if $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = \{e\}$ or G is abelian.

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* *Hint:* First, verify that $(xy)^{mn} = e$. Now, suppose $|xy| = k$. What do you immediately know about the relationship between k and mn. Next, consider $(xy)^{kn}$. Argue that m divides kn and then argue that m divides k. Similarly, n divides k. Ultimately, conclude that $mn = k$.

† *Hint:* Suppose $g \in G$ such that $|g| = n$. Let h be an arbitrary element in G such that $|h| = m$. You need to show that m divides n. For sake of a contradiction, assume otherwise. Then there exists a prime p whose multiplicity as a factor of m exceeds that of n. Let p^a be the highest power of p in m and p^b be the highest power of p in n, so $a > b$. Consider the elements g^{p^b} and h^{m/p^a}.

‡ *Hint:* Let n be the maximal order among the elements of G and let $g \in G$ be an element with order n. Prove that $G = \langle g \rangle$.

§ This is Theorem 7.11. You may use Theorem 5.35 if you'd like, but it isn't required.

¶ This is one direction of Theorem 7.13. It turns out that the converse is also true, but you do not need to worry about proving it.

