MAT 411: Introduction to Abstract Algebra
Final Exam (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of the Final Exam is worth a total of 12 points and is worth 20% of your overall score on the Final. This take-home exam is due by 9AM on **Friday, December 13**. Your overall score on the Final Exam is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 1.41, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

 Written by D.C. Ernst

On this exam, you may use a the result of an earlier problem in the proof of a later problem (even if you did not complete the earlier problem). If a problem below appears in Chapter 8 of the course notes, you are free to use any result in Chapter 8 that appears prior to the problem in question. Otherwise, you are free to use any result in Chapter 8.

Problem 1. (4 points) Prove **one** of the following.

(a) Any finite integral domain is a field.*

(b) Assume R is a commutative ring with $1 \neq 0$. Let I be an ideal of R. Then $I = R$ if and only if I contains a unit.†

(c) Assume R is a commutative ring with $1 \neq 0$. Then R is a field if and only if its only ideals are (0) and R.‡

(d) If R is a field, then every nonzero ring homomorphism from R into another ring is an injection.§

Problem 2. (4 points) Prove **one** of the following.

(a) Assume R is a commutative ring with 1. Then M is a maximal ideal if and only if the quotient ring R/M is a field.¶

(b) Assume R is a commutative ring with 1. Then P is a prime ideal in R if and only if the quotient ring R/P is an integral domain.‖

Problem 3. (4 points) Prove **one** of the following.

(a) As rings, $\mathbb{Z}[x]/(x-2) \cong \mathbb{Z}$.**

(b) Let R be commutative ring with 1. Then the principal ideal (x) in $R[x]$ is a maximal ideal if and only if R is a field.††

(c) Define $\phi : \mathbb{Z}_{10} \to \mathbb{Z}_{10}$ via $\phi(x) = 6x$. Then ϕ is a ring homomorphism, $\phi(\mathbb{Z}_{10})$ is a field, and $\ker(\phi)$ is a maximal ideal of $\mathbb{Z}_{10}$.‡‡

*This is Theorem 8.25. *Hint:* Let R be a finite integral domain. Then among other things, R has a 1, say 1_R. Let $a \in R \setminus \{0\}$ and define $\phi_a : R \to R$ via $\phi_a(r) = ar$. Prove some useful properties about ϕ_a and cleverly use this function to prove that a has a multiplicative inverse.

†This is Theorem 8.49.

‡This is Theorem 8.50.

§This is Corollary 8.51.

¶This is Theorem 8.55. For the forward implication, try using a proof by contradiction. Assume that R/M is not a field. Then by one of the theorems in Problem 1, there exists an ideal J/M of R/M, where J is a subring of R such that $M \subset J \subset R$ $(J \neq M, R)$. Now, let $r \in R$ and $j \in J$. Compute $(r+M)(j+M)$ and using the fact that J/M is an ideal, prove that J is an ideal, which contradicts M being maximal.

‖This is Theorem 8.60.

**Try using the First Isomorphism Theorem for Rings.

††Try using the First Isomorphism Theorem for Rings to first prove that $R[x]/(x) \cong R$.

‡‡Notice that there are three things to prove. You must prove the first of these three things before the other two, but you can prove the remaining two facts in either order.

