Problem 6.33. Find the order of the given element in the quotient group. You may assume that we are taking the quotient by a normal subgroup.

(a) $s\langle r\rangle \in D_4/\langle r\rangle$

(b) $j\langle -1\rangle \in Q_8/\langle -1\rangle$

(c) $5+\langle 4\rangle \in \mathbb{Z}_{12}/\langle 4\rangle$

(d) $(2,1)+\langle(1,1)\rangle \in (\mathbb{Z}_3\times\mathbb{Z}_6)/\langle(1,1)\rangle$

(e) $(1,3)+\langle(0,2)\rangle \in (\mathbb{Z}_4\times\mathbb{Z}_8)/\langle(0,2)\rangle$

Note: The order of gH in G/H is the smallest positive exponent s.t. $(gH)^k = H$ (recall that H is the identity in G/H). This is equivalent to the smallest positive exponent k s.t. $g^k \in H$.

(a) Note that $\langle r\rangle = \{e, r, r^2, r^3\}$. Since $s \notin \langle r\rangle$ yet $s^2 = e \in \langle r\rangle$, $|s\langle r\rangle| = 2$.

(b) Note that $\langle -1\rangle = \{1, -1\}$. Since $j \notin \langle -1\rangle$ yet $j^2 = -1 \in \langle -1\rangle$, $|j\langle -1\rangle| = 2$.

(c) Note that $\langle 4\rangle = \{0, 4, 8\}$. Since $5 \notin \langle 4\rangle$, $5+5 = 10 \notin \langle 4\rangle$, $5+5+5 = 3 \notin \langle 4\rangle$, yet $5+5+5+5 = 8 \in \langle 4\rangle$, $|5+\langle 4\rangle| = 4$.

(d) Note that

$$\langle(1,1)\rangle = \{(0,0), (1,1), (2,2), (0,3), (1,4), (2,5)\}.$$

Since $(2,1) \notin \langle(1,1)\rangle$,

$$(2,1)+(2,1) = (1,2) \notin \langle(1,1)\rangle$$

$$(2,1)+(2,1)+(2,1) = (0,3) \in \langle(1,1)\rangle,$$

$$|(2,1)+\langle(1,1)\rangle| = 3.$$

(e) Note that

$$\langle(0,2)\rangle = \{(0,0), (0,2), (0,4), (0,6)\}.$$

Since

$$(1,3) \notin \langle(0,2)\rangle,$$

$$(1,3)+(1,3) = (2,6) \notin \langle(0,2)\rangle,$$

$$(1,3)+(1,3)+(1,3) = (3,1) \notin \langle(0,2)\rangle,$$

$$(1,3)+(1,3)+(1,3)+(1,3) = (0,4) \in \langle(0,2)\rangle,$$

$$|(1,3)+\langle(0,2)\rangle| = 4.$$