Problem 6.34. For each quotient group below, describe the group. If possible, state what group each is isomorphic to. You may assume that we are taking the quotient by a normal subgroup.

(a) $Q_8/\langle -1\rangle$

(b) $Q_8/\langle i\rangle$

(c) $\mathbb{Z}_4/\langle 2\rangle$

(d) $V_4/\langle h\rangle$

(e) $A_4/\langle (1,2)(3,4),(1,3)(2,4)\rangle$

(f) $(\mathbb{Z}_2\times\mathbb{Z}_2)/\langle (1,1)\rangle$

(g) $\mathbb{Z}/4\mathbb{Z}$

(h) S_4/A_4

(i) $(\mathbb{Z}_4\times\mathbb{Z}_2)/(\{0\}\times\mathbb{Z}_2)$

<u>Note</u>: Recall that $|G/H| =$ # of cosets of H in G. When G is finite, this is simply

$$|G/H| = \frac{|G|}{|H|}.$$

However, G/H may be finite even if G is not. For many of the problems above, we can quickly determine the isomorphism type from the order alone. In particular, if the order is prime, say p, then we know the group is isomorphic to $\mathbb{Z}_p$.

If the order is not prime, we might need to do more work.

Here are the easy ones:

(b) $\left|Q_8/\langle i\rangle\right| = \frac{8}{4} = 2 \Rightarrow Q_8/\langle i\rangle \cong \mathbb{Z}_2$

(c) $\left|\mathbb{Z}_4/\langle 2\rangle\right| = \frac{4}{2} = 2 \Rightarrow \mathbb{Z}_4/\langle 2\rangle \cong \mathbb{Z}_2$

(d) $\left|V_4/\langle h\rangle\right| = \frac{4}{2} = 2 \Rightarrow V_4/\langle h\rangle \cong \mathbb{Z}_2$

(e) Recall that $|A_4| = 12$. It's not too hard to verify that

$\langle (12)(34), (13)(24)\rangle = \{e, (12)(34), (13)(24), (14)(23)\}$.

This implies that $\left|A_4/\langle (12)(34),(13)(24)\rangle\right| = 3$.

Thus, $A_4/\langle (12)(34),(13)(24)\rangle \cong \mathbb{Z}_3$.

(f) $\left|\mathbb{Z}_2\times\mathbb{Z}_2/\langle(1,1)\rangle\right| = \frac{4}{2} = 2$

$\Rightarrow \mathbb{Z}_2\times\mathbb{Z}_2/\langle(1,1)\rangle \cong \mathbb{Z}_2$

(h) $|S_n/A_n| = \frac{n!}{n!/2} = 2 \Rightarrow S_n/A_n \cong \mathbb{Z}_2$.

The harder ones are (a), (g), (i).

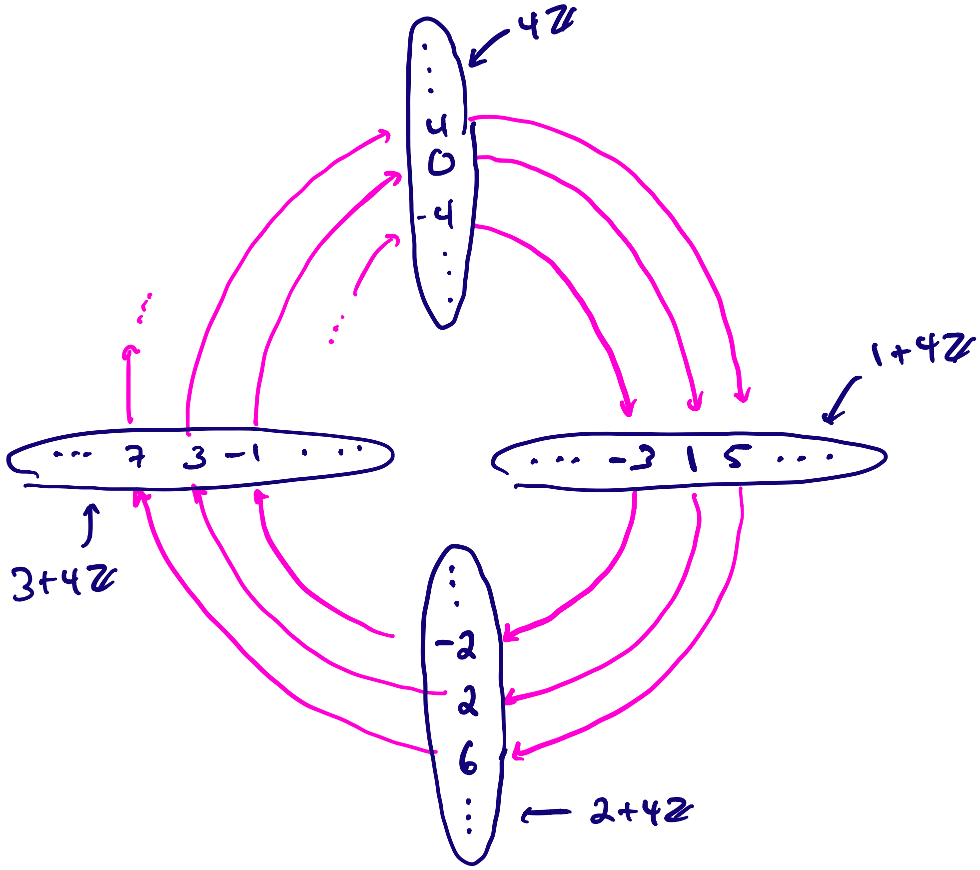
(a) We've done this one a couple times. In particular, see Figure 6.2.

$$Q_8/\langle -1 \rangle \cong V_4.$$

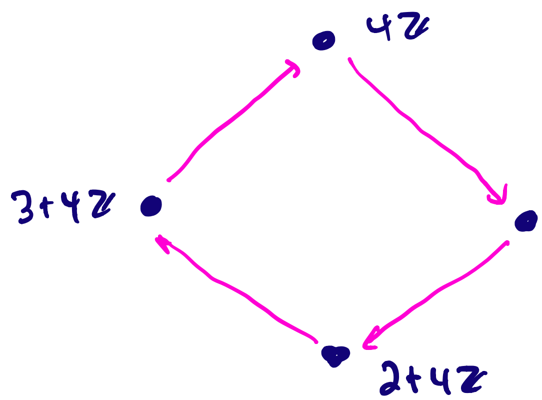
(g) First, observe that

$\mathbb{Z}/4\mathbb{Z} = \{4\mathbb{Z}, 1+4\mathbb{Z}, 2+4\mathbb{Z}, 3+4\mathbb{Z}\}$,

so that $|\mathbb{Z}/4\mathbb{Z}| = 4$. This implies that $\mathbb{Z}/4\mathbb{Z}$ is isomorphic to either $\mathbb{Z}_4$ or V_4. However, since $1, 1+1=2, 1+1+1=3 \notin 4\mathbb{Z}$ yet $1+1+1+1=0$ is in $4\mathbb{Z}$, $|1+4\mathbb{Z}| = 4$, which implies that $\langle 1+4\mathbb{Z} \rangle = \mathbb{Z}/4\mathbb{Z}$, and hence $\mathbb{Z}/4\mathbb{Z}$ is cyclic. Thus, $\mathbb{Z}/4\mathbb{Z} \cong \mathbb{Z}_4$. We can also do the quotient process to see that this is true:



$\downarrow$ quotient process



$\cong \mathbb{Z}_4$

(i) First, observe that $|\mathbb{Z}_4 \times \mathbb{Z}_2| = 8$ while $|\{0\} \times \mathbb{Z}_2| = 2$. Then

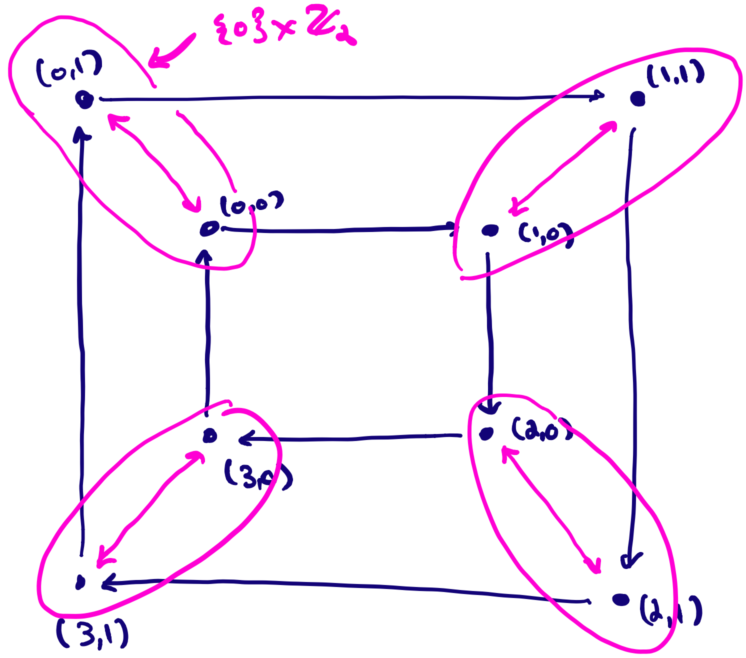
$$\left| \mathbb{Z}_4 \times \mathbb{Z}_2 / \{0\} \times \mathbb{Z}_2 \right| = 4,$$

and so $\mathbb{Z}_4 \times \mathbb{Z}_2 / \{0\} \times \mathbb{Z}_2$ is either iso to $\mathbb{Z}_4$ or V_4. There are two possible approaches to determine the correct ans:

(1) Compute the orders of elmts in the quotient. If there is an elmt of order 4, then the ans is $\mathbb{Z}_4$; o/w its V_4.

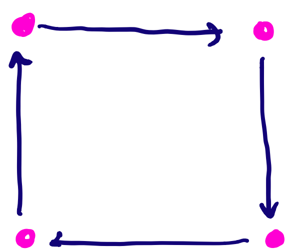
(2) Do the quotient process to the cayley diagram.

Pictures are fun, so let's go w/ 2nd option.



$$\mathbb{Z}_4 \times \mathbb{Z}_2 = \langle (1,0), (0,1) \rangle$$

$\big\downarrow$ quotient process



$$\cong \mathbb{Z}_4$$

$$\mathbb{Z}_4 \times \mathbb{Z}_2 / \{0\} \times \mathbb{Z}_2$$