Theorem 6.20. Suppose G_1 and G_2 are groups such that $H_1 \leq G_1$ and $H_2 \leq G_2$. Then $H_1 \times H_2 \leq G_1 \times G_2$.

Pf: Suppose $H_1 \leq G_1$, $H_2 \leq G_2$. We will utilize the 2-step subgrp test.

(0) Since $e_1 \in H_1$ and $e_2 \in H_2$ (where e_1 and e_2 are the respective identities), $(e_1, e_2) \in H_1 \times H_2$, and hence $H_1 \times H_2 \neq \phi$.

(1) Let $(x_1, y_1), (x_2, y_2) \in H_1 \times H_2$. Then $x_1, x_2 \in H_1$ and $y_1, y_2 \in H_2$. Since H_1 and H_2 are closed, $x_1 x_2 \in H_1$ and $y_1 y_2 \in H_2$. This implies that

$$(x_1, y_2)(x_2, y_2) = (x_1 x_2, y_1 y_2) \in H_1 \times H_2.$$

Thus, $H_1 \times H_2$ is closed.

(2) Now, let $(x, y) \in H_1 \times H_2$. Then $x \in H_1$ and $y \in H_2$. Since H_1 and H_2 are grps, $x^{-1} \in H_1$ and $y^{-1} \in H_2$. This shows that

$$(x, y)^{-1} = (x^{-1}, y^{-1}) \in H_1 \times H_2.$$

$\blacksquare$