If you want to sharpen a sword, you have to remove a little metal.

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Chapter 8

Differentiation

It's time for derivatives!

8.1 Introduction to Differentiation

Definition 8.1. Let $f : A \to \mathbb{R}$ be a real function and let $a \in A$ such that f is defined on some open interval I containing a (i.e., $a \in I \subseteq A$). The **derivative** of f at a is defined via

$$\boxed{f'(a) := \lim_{x \to a} \frac{f(x) - f(a)}{x - a}}$$

provided this limit exists. If $f'(a)$ exists, then we say that f is **differentiable** at a. More generally, we say that f is **differentiable** on $B \subseteq A$ if f is differentiable at every point in B. As a special case, f is said to be **differentiable** if it is differentiable at every point in its domain. If f does indeed have a derivative at some points in its domain, then the **derivative** of f is the function denoted by f', such that for each number x at which f is differentiable, $f'(x)$ is the derivative of f at x. We may also write

$$\boxed{\frac{d}{dx}[f(x)] := f'(x).}$$

The lefthand side of the equation above is typically read as, "the derivative of f with respect to x." The notation $f'(x)$ is commonly referred to as "Newton's notation" for the derivative while $\frac{d}{dx}[f(x)]$ is often referred to as "Liebniz's notation".

Note that the definition of the derivative automatically excludes the kind of behavior we saw with continuous functions, where a function defined only at a single point was continuous.

Problem 8.2. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 - x + 1$ at $a = 2$. Prove that f is differentiable on $\mathbb{R}$ and find a formula for the derivative of f.

Problem 8.3. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = c$ for some constant $c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and $f'(x) = 0$ for all $x \in \mathbb{R}$.

Problem 8.4. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = mx + b$, where $m, b \in \mathbb{R}$. Prove that f is differentiable and $f'(x) = m$ for all $x \in \mathbb{R}$.

Problem 8.5. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = ax^2 + bx + c$, where $a, b, c \in \mathbb{R}$. Prove that f is differentiable on $\mathbb{R}$ and find a formula for the derivative of f.

In the previous four problems, note that if we restrict the domain of the functions to a closed interval $[a, b]$, then we can conclude that we get the expected derivatives for all $x \in (a, b)$.

Problem 8.6. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^3$. Prove that f is differentiable at 0 and $f'(0) = 0$.

Problem 8.7. Explain why any function defined only on $\mathbb{Z}$ cannot have a derivative.

The next problem tells us that differentiability implies continuity.

Problem 8.8. Prove that if f has a derivative at $x = a$, then f is also continuous at $x = a$.

The converse of the previous theorem is not true. That is, continuity does not imply differentiability.

Problem 8.9. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = |x|$.

(a) Prove that f is continuous at every point in its domain.

(b) Prove that f is differentiable everywhere except at $x = 0$.

Problem 8.10. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} x, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Show that f is continuous at $x = 0$, but not differentiable at $x = 0$.

> Don't let anyone rob you of your imagination, your creativity, or your curiosity. It's your place in the world; it's your life. Go on and do all you can with it, and make it the life you want to live.
>
> Mae Jemison, NASA astronaut

8.2 Derivative Rules

In this section, we prove a few of the well-known derivative rules from first-semester calculus.

Problem 8.11. If f is differentiable at x and $c \in \mathbb{R}$, prove that the function cf also has a derivative at x and $(cf)'(x) = cf'(x)$.

Problem 8.12. If f and g are differentiable at x, show that the function $f+g$ also has a derivative at x and $(f+g)'(x) = f'(x) + g'(x)$.

The next problem states the well-known Product and Quotient Rules for Derivatives. You will need to use Problem 8.8 in their proofs.

Problem 8.13. Suppose f and g are differentiable at x. Prove each of the following:

(a) (Product Rule) The function fg is differentiable at x. Moreover, its derivative function is given by
$$(fg)'(x) = f'(x)g(x) + f(x)g'(x).$$

(b) (Quotient Rule) The function f/g is differentiable at x provided $g(x) \neq 0$. Moreover, its derivative function is given by
$$\left(\frac{f}{g}\right)'(x) = \frac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}.$$

The next problem is sure to make your head hurt.

Problem 8.14. Define $g : \mathbb{R} \to \mathbb{R}$ via
$$g(x) = \begin{cases} 0, & \text{if } x \in \mathbb{Q} \\ 1, & \text{otherwise.} \end{cases}$$
Now, define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2 g(x)$. Determine where f is differentiable.

> I write one page of masterpiece to ninety-one pages of shit.
>
> Ernest Hemingway, novelist & journalist

8.3 The Mean Value Theorem

The next result tells us that if a differentiable function attains a maximum value at some point in an open interval contained in the domain of the function, then the derivative is zero at that point. In a calculus class, we would say that differentiable functions attain local maximums at critical numbers.

Problem 8.15. Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$, $f'(c)$ exists for some $c \in (a,b)$, and $f(c) \geq f(x)$ for all $x \in (a,b)$. Prove that $f'(c) = 0$.

Problem 8.16. Let $f : A \to \mathbb{R}$ be a real function such that $f'(c) = 0$ for some $c \in A$. Does this imply that there exists an open interval $(a,b) \subseteq A$ containing c such that either $f(x) \geq f(c)$ or $f(x) \leq f(c)$ for all $x \in (a,b)$? If so, prove it. Otherwise, provide a counterexample.

The next problem asks you to prove a result called Rolle's Theorem. To prove Rolle's Theorem, first, apply the Extreme Value Theorem to f and $-f$ to conclude that f attains both a maximum and minimum on $[a,b]$. If both the maximum and minimum are attained at the end points of $[a,b]$, then the maximum and minimum are the same, and hence the function is constant. What does Problem 8.3 tell us in this case? But what if f is not constant over $[a,b]$? Try using Problem 8.15.

Problem 8.17 (Rolle's Theorem)**.** Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$, differentiable on (a,b), and $f(a) = f(b)$, then prove that there exists a point $c \in (a,b)$ such that $f'(c) = 0$.

We can use Rolle's Theorem to prove the next result, which is the well-known Mean Value Theorem.

Problem 8.18 (Mean Value Theorem)**.** Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b), then prove that there exists a point $c \in (a,b)$ such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}.$$

Here's one approach. Cleverly define the function $g(x) = f(x) - \frac{f(b)-f(a)}{b-a}(x-a)$. Is g continuous on $[a,b]$? Is g differentiable on (a,b)? Can we apply Rolle's Theorem to g using the interval $[a,b]$? What can you conclude? Magic!

Problem 8.19. Let $f : A \to \mathbb{R}$ be a real function such that $[a,b] \subseteq A$. If f is continuous on $[a,b]$ and differentiable on (a,b) such that $f'(x) = 0$ for all $x \in (a,b)$, then prove that f is constant over $[a,b]$. Try applying the Mean Value Theorem to $[a,t]$ for every $t \in (a,b]$.

Problem 8.20. Let $f : A \to \mathbb{R}$ be a real function such that $(a,b) \subseteq A$ and f is differentiable at every point of (a,b). Prove that if $f'(x) > 0$ for all $x \in (a,b)$, then f is strictly increasing on (a,b). What if $f'(x) < 0$ for all $x \in (a,b)$?

The converse of the previous result is not true in general!

Problem 8.21. Find an example of a function $f : \mathbb{R} \to \mathbb{R}$ such that f is strictly increasing on its domain yet $f'(x)$ is not positive for all $x \in \mathbb{R}$.

Problem 8.22. Let $f : A \to \mathbb{R}$ and $g : A \to \mathbb{R}$ such that $[a,b] \subseteq A$. Prove that if f and g are continuous on $[a,b]$ and $f'(x) = g'(x)$ for all $x \in (a,b)$, then there exists $C \in \mathbb{R}$ such that $f(x) = g(x) + C$ for all $x \in [a,b]$.

Problem 8.23. Is the converse of the previous problem true? If so, prove it. Otherwise, provide a counterexample.

Problem 8.24. Let $f : A \to \mathbb{R}$ be a real function such that $(a,b) \subseteq A$ and f is differentiable at every point of (a,b). Prove that if there exists a nonnegative real number M such that $|f'(x)| \leq M$ for all $x \in (a,b)$ (i.e., f has bounded derivative on (a,b)), then for $x, y \in (a,b)$, we have $|f(x) - f(y)| \leq M|x-y|$.

Problem 8.25. Let $f : A \to \mathbb{R}$ be a real function such that $(a,b) \subseteq A$ and f is differentiable at every point of (a,b). Prove that if there exists a nonnegative real number M such that $|f'(x)| \leq M$ for all $x \in (a,b)$ (i.e., f has bounded derivative on (a,b)), then f is uniformly continuous on (a,b).

Problem 8.26. If f is a differentiable real function that is also uniformly continuous, does this imply that the derivative is bounded? Justify your assertion.

> It is not the critic who counts; not the man who points out how the strong man stumbles, or where the doer of deeds could have done them better. The credit belongs to the man who is actually in the arena, whose face is marred by dust and sweat and blood; who strives valiantly; who errs, who comes short again and again, because there is no effort without error and shortcoming; but who does actually strive to do the deeds; who knows great enthusiasms, the great devotions; who spends himself in a worthy cause; who at the best knows in the end the triumph of high achievement, and who at the worst, if he fails, at least fails while daring greatly, so that his place shall never be with those cold and timid souls who neither know victory nor defeat.
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> Theodore Roosevelt, statesman & conservationist