Pure mathematics is the poetry of logical ideas.

Albert Einstein, theoretical physicist

Chapter 2

Preliminaries

In this chapter, we summarize some background material we need to be familiar with.

2.1 Sets

At its essence, all of mathematics is built on set theory.

A **set** is a collection of objects called **elements**. If A is a set and x is an element of A, we write $\boxed{x \in A}$. Otherwise, we write $\boxed{x \notin A}$. The set containing no elements is called the **empty set**, and is denoted by the symbol $\boxed{\emptyset}$. Any set that contains at least one element is referred to as a **nonempty set**.

If we think of a set as a box potentially containing some stuff, then the empty set is a box with nothing in it. One assumption we will make is that for any set A, $A \notin A$. The language associated to sets is specific. We will often define sets using the following notation, called **set-builder notation**:

$$\boxed{S = \{x \in A \mid P(x)\}},$$

where $P(x)$ is some predicate statement involving x. The first part "$x \in A$" denotes what type of x is being considered. The predicate to the right of the vertical bar (not to be confused with "divides") determines the condition(s) that each x must satisfy in order to be a member of the set. This notation is read as "The set of all x in A such that $P(x)$." As an example, the set $\{x \in \mathbb{N} \mid x \text{ is even and } x \geq 8\}$ describes the collection of even natural numbers that are greater than or equal to 8.

There are a few sets that are commonly discussed in mathematics and have predefined symbols to denote them. We've already encountered the integers, natural numbers, and real numbers. Notice that our definition of the rational numbers uses set-builder notation.

- **Natural numbers:** $\boxed{\mathbb{N} := \{1, 2, 3, \ldots\}}$. Some books will include zero in the set of natural numbers, but we do not.
- **Integers:** $\boxed{\mathbb{Z} := \{0, \pm 1, \pm 2, \pm 3, \ldots\}}$.
- **Rational Numbers:** $\boxed{\mathbb{Q} := \{a/b \mid a, b \in \mathbb{Z} \text{ and } b \neq 0\}}$.

- **Real Numbers:** $\boxed{\mathbb{R}}$ denotes the set of real numbers.

Since the set of natural numbers consists of the positive integers, the natural numbers are sometimes denoted by $\boxed{\mathbb{Z}^+}$.

If A and B are sets, then we say that A is a **subset** of B, written $\boxed{A \subseteq B}$, provided that every element of A is an element of B. Observe that $A \subseteq B$ is equivalent to "For all x in the universe of discourse, if $x \in A$, then $x \in B$."

Every nonempty set always has two rather boring subsets.

Problem 2.1. Let A be a set. Write a short proof for each of the following.

(a) $A \subseteq A$

(b) $\emptyset \subseteq A$

The next problem shows that "$\subseteq$" is a transitive relation.

Problem 2.2 (Transitivity of subsets)**.** Prove that if A, B, and C are sets such that $A \subseteq B$ and $B \subseteq C$, then $A \subseteq C$.

Let A and B be sets in some universe of discourse U. We define the following.

- Two sets A and B are **equal**, denoted $\boxed{A = B}$, if the sets contain the same elements. That is, $A = B$ if and only if $A \subseteq B$ and $B \subseteq A$. Note that if we want to prove $A = B$, then we have to do two separate subproofs: one for $A \subseteq B$ and one for $B \subseteq A$. It is common to label each mini-proof with "($\subseteq$)" and "($\supseteq$)", respectively.

- If $A \subseteq B$, then A is called a **proper subset** provided that $A \neq B$. In this case, we may write $\boxed{A \subset B}$ or $\boxed{A \subsetneq B}$. *Warning:* Some books use $\subset$ to mean $\subseteq$.

- The **union** of the sets A and B is $\boxed{A \cup B := \{x \in U \mid x \in A \text{ or } x \in B\}}$.

- The **intersection** of the sets A and B is $\boxed{A \cap B := \{x \in U \mid x \in A \text{ and } x \in B\}}$.

- The **set difference** of the sets A and B is $\boxed{A \setminus B := \{x \in U \mid x \in A \text{ and } x \notin B\}}$.

- The **complement of** A (relative to U) is the set $\boxed{A^c := U \setminus A = \{x \in U \mid x \notin A\}}$.

- If $A \cap B = \emptyset$, then we say that A and B are **disjoint** sets.

Example 2.3. The set $\mathbb{R} \setminus \mathbb{Q}$ is called the set of **irrational numbers**.

Problem 2.4. Prove that if A and B are sets such that $A \subseteq B$, then $B^c \subseteq A^c$.

Problem 2.5. Prove that if A and B are sets, then $A \setminus B = A \cap B^c$.

Problem 2.6. Give an example where $A \neq B$ but $A \setminus B = \emptyset$.

Consider the following collection of sets:

$$\{a\}, \{a,b\}, \{a,b,c\}, \ldots, \{a,b,c,\ldots,z\}$$

This collection has a natural way for us to "index" the sets:

$$A_1 = \{a\}, A_2 = \{a,b\}, A_3 = \{a,b,c\}, \ldots, A_{26} = \{a,b,c,\ldots,z\}$$

In this case the sets are **indexed** by the set $\{1,2,\ldots,26\}$, where the subscripts are taken from the **index set**. If we wanted to talk about an arbitrary set from this indexed collection, we could use the notation A_n.

Using indexing sets in mathematics is an extremely useful notational tool, but it is important to keep straight the difference between the sets that are being indexed, the elements in each set being indexed, the indexing set, and the elements of the indexing set.

Any set (finite or infinite) can be used as an indexing set. Often capital Greek letters are used to denote arbitrary indexing sets and small Greek letters to represent elements of these sets. If the indexing set is a subset of $\mathbb{R}$, then it is common to use Roman letters as individual indices. Of course, these are merely conventions, not rules.

- If Δ is a set and we have a collection of sets indexed by Δ, then we may write $\{S_\alpha\}_{\alpha\in\Delta}$ to refer to this collection. We read this as "the set of S-sub-alphas over alpha in Delta."
- If a collection of sets is indexed by $\mathbb{N}$, then we may write $\{U_n\}_{n\in\mathbb{N}}$ or $\{U_n\}_{n=1}^{\infty}$.
- Borrowing from this idea, a collection $\{A_1,\ldots,A_{26}\}$ may be written as $\{A_n\}_{n=1}^{26}$.

Suppose we have a collection $\{A_\alpha\}_{\alpha\in\Delta}$.

- The **union of the entire collection** is defined via

$$\boxed{\bigcup_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for some } \alpha \in \Delta\}.}$$

- The **intersection of the entire collection** is defined via

$$\boxed{\bigcap_{\alpha\in\Delta} A_\alpha = \{x \mid x \in A_\alpha \text{ for all } \alpha \in \Delta\}.}$$

In the special case that $\Delta = \mathbb{N}$, we write

$$\bigcup_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for some } n \in \mathbb{N}\} = A_1 \cup A_2 \cup A_3 \cup \cdots$$

and

$$\bigcap_{n=1}^{\infty} A_n = \{x \mid x \in A_n \text{ for all } n \in \mathbb{N}\} = A_1 \cap A_2 \cap A_3 \cap \cdots$$

Similarly, if $\Delta = \{1,2,3,4\}$, then

$$\bigcup_{n=1}^{4} A_n = A_1 \cup A_2 \cup A_3 \cup A_4 \quad \text{and} \quad \bigcap_{n=1}^{4} A_n = A_1 \cap A_2 \cap A_3 \cap A_4.$$

Notice the difference between "$\bigcup$" and "$\cup$" (respectively, "$\bigcap$" and "$\cap$").

Problem 2.7. Let $\{A_n\}_{n=1}^{26}$ be the collection from the discussion below Problem 2.6. Find each of the following.

(a) $\displaystyle\bigcup_{n=1}^{26} A_n$ (b) $\displaystyle\bigcap_{n=1}^{26} A_n$

Problem 2.8. For each $r \in \mathbb{Q}$ (the rational numbers), let N_r be the set containing all real numbers *except* r. Find each of the following.

(a) $\displaystyle\bigcup_{r\in\mathbb{Q}} N_r$ (b) $\displaystyle\bigcap_{r\in\mathbb{Q}} N_r$

A collection of sets $\{A_\alpha\}_{\alpha\in\Delta}$ is **pairwise disjoint** if $A_\alpha \cap A_\beta = \emptyset$ for $\alpha \neq \beta$.

Problem 2.9. Draw a Venn diagram of a collection of three sets that are pairwise disjoint.

Problem 2.10. Provide an example of a collection of three sets, say $\{A_1, A_2, A_3\}$, such that the collection is *not* pairwise disjoint, but $\bigcap_{n=1}^{3} A_n = \emptyset$.

Problem 2.11. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_{i+1} \subsetneq S_i$ and $\bigcap_{i=1}^{\infty} S_i = \emptyset$.

Problem 2.12. Find a collection of nonempty sets $S_i \subseteq \mathbb{N}$ indexed by $i \in \mathbb{N}$ such that $S_i \subsetneq S_{i+1}$ but $\bigcup_{i=1}^{\infty} S_i \neq \mathbb{N}$.

Problem 2.13 (DeMorgan's Law)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets. Prove one of the following.

(a) $\displaystyle\left(\bigcup_{\alpha\in\Delta} A_\alpha\right)^C = \bigcap_{\alpha\in\Delta} A_\alpha^C$ (b) $\displaystyle\left(\bigcap_{\alpha\in\Delta} A_\alpha\right)^C = \bigcup_{\alpha\in\Delta} A_\alpha^C$

Problem 2.14 (Distribution of Union and Intersection)**.** Let $\{A_\alpha\}_{\alpha\in\Delta}$ be a collection of sets and let B be any set. Prove one of the following.

(a) $\displaystyle B \cup \left(\bigcap_{\alpha\in\Delta} A_\alpha\right) = \bigcap_{\alpha\in\Delta} (B \cup A_\alpha)$ (b) $\displaystyle B \cap \left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} (B \cap A_\alpha)$

An **ordered pair** is an ordered list of two elements of the form (a,b). In this case, a is called the **first component** (or **first coordinate**) while b is called the **second component** (or **second coordinate**). We can use the notion of ordered pairs to construct new sets from existing sets. If A and B are sets, the **Cartesian product** (or **direct product**) of A and B, denoted $A \times B$ (read as "A times B" or "A cross B"), is the set of all ordered pairs where the first component is from A and the second component is from B. In set-builder notation, we have

$$\boxed{A \times B := \{(a,b) \mid a \in A, b \in B\}}.$$

Example 2.15. The standard two-dimensional plane $\mathbb{R}^2$ is a familiar example of Cartesian product. In particular, we have

$$\mathbb{R}^2 = \mathbb{R} \times \mathbb{R} = \{(x,y) \mid x, y \in \mathbb{R}\}.$$

Problem 2.16. If A is a set, then what is $A \times \emptyset$ equal to?

Problem 2.17. Given sets A and B, when will $A \times B$ be equal to $B \times A$?

> It does not matter how slowly you go as long as you do not stop.
>
> Confucius, philosopher

2.2 Induction and The Well-Ordering Principle

The following axiom is one of the Peano Axioms, which is a collection of axioms for the natural numbers introduced in the 19th century by Italian mathematician Giuseppe Peano (1858–1932).

Axiom 2.18 (Axiom of Induction)**.** Let $S \subseteq \mathbb{N}$ such that both

(i) $1 \in S$, and

(ii) if $k \in S$, then $k+1 \in S$.

Then $S = \mathbb{N}$.

We can think of the set S as a ladder, where the first hypothesis as saying that we have a first rung of a ladder. The second hypothesis says that if we are on any arbitrary rung of the ladder, then we can always get to the next rung. Taken together, this says that we can get from the first rung to the second, from the second to the third, and in general, from any kth rung to the $(k+1)$st rung, so that our ladder is actually $\mathbb{N}$. Do you agree that the Axiom of Induction is a pretty reasonable assumption?

Using the Axiom of Induction, we can prove the following theorem, known as the **Principle of Mathematical Induction**. One approach to proving this theorem is to let $S = \{k \in \mathbb{N} \mid P(k) \text{ is true}\}$ and use the Axiom of Induction. The set S is sometimes called the **truth set**. Your job is to show that the truth set is all of $\mathbb{N}$.

Problem 2.19 (Principle of Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume

(i) $P(1)$ is true, and

(ii) if $P(k)$ is true, then $P(k+1)$ is true.

Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

The Principle of Mathematical Induction provides us with a process for proving statements of the form: "For all $n \in \mathbb{N}$, $P(n)$," where $P(n)$ is some predicate involving n. Hypothesis (i) above is called the **base step** (or **base case**) while (ii) is called the **inductive step**.

You should not confuse *mathematical induction* with *inductive reasoning* associated with the natural sciences. Inductive reasoning is a scientific method whereby one induces general principles from observations. On the other hand, mathematical induction is a deductive form of reasoning used to establish the validity of a proposition.

There is another formulation of induction, where the inductive step begins with a set of assumptions rather than one single assumption. This method is sometimes called **complete induction** (or **strong induction**).

Problem 2.20 (Principle of Complete Mathematical Induction)**.** Let $P(1), P(2), P(3), \ldots$ be a sequence of statements, one for each natural number. Assume that

(i) $P(1)$ is true, and

(ii) For all $k \in \mathbb{N}$, if $P(j)$ is true for all $j \in \mathbb{N}$ such that $j \leq k$, then $P(k+1)$ is true.

Prove that $P(n)$ is true for all $n \in \mathbb{N}$.

Note the difference between ordinary induction and complete induction. For the induction step of complete induction, we are not only assuming that $P(k)$ is true, but rather that $P(j)$ is true for all j from 1 to k. Despite the name, complete induction is not any stronger or more powerful than ordinary induction. It is worth pointing out that anytime ordinary induction is an appropriate proof technique, so is complete induction. So, when should we use complete induction?

In the inductive step, you need to reach $P(k+1)$, and you should ask yourself which of the previous cases you need to get there. If all you need, is the statement $P(k)$, then ordinary induction is the way to go. If two preceding cases, $P(k-1)$ and $P(k)$, are necessary to reach $P(k+1)$, then complete induction is appropriate. In the extreme, if one needs the full range of preceding cases (i.e., all statements $P(1), P(2), \ldots, P(k)$), then again complete induction should be utilized.

Note that in situations where complete induction is appropriate, it might be the case that you need to verify more than one case in the base step. The number of base cases to be checked depends on how one needs to "look back" in the induction step.

The penultimate result of this section is known as the **Well-Ordering Principle**. This seemingly obvious theorem requires a bit of work to prove. It is worth noting that in some axiomatic systems, the Well-Ordering Principle is sometimes taken as an axiom. However, in our case, we will assume the Axiom of Induction and then prove the result using complete induction. Before stating the Well-Ordering Principle, we need an additional definition.

Definition 2.21. Let $A \subseteq \mathbb{R}$ and $m \in A$. Then m is called a **maximum** (or **greatest element**) of A if for all $a \in A$, we have $a \leq m$. Similarly, m is called **minimum** (or **least element**) of A if for all $a \in A$, we have $m \leq a$.

Not surprisingly, maximums and minimums are unique when they exist.

Problem 2.22. Prove that if $A \subseteq \mathbb{R}$ such that the maximum (respectively, minimum) of A exists, then the maximum (respectively, minimum) of A is unique.

If the maximum of a set A exists, then it is denoted by $\boxed{\max(A)}$. Similarly, if the minimum of a set A exists, then it is denoted by $\boxed{\min(A)}$.

Problem 2.23. Find the maximum and the minimum for each of the following sets when they exist.

(a) $\{5, 11, 17, 42, 103\}$

(b) $\mathbb{N}$

(c) $\mathbb{Z}$

(d) $(0, 1]$

(e) $(0, 1] \cap \mathbb{Q}$

(f) $(0, \infty)$

(g) $\{42\}$

(h) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(i) $\{\frac{1}{n} \mid n \in \mathbb{N}\} \cup \{0\}$

(j) $\emptyset$

To prove the Well-Ordering Principle, consider a proof by contradiction. Suppose S is a nonempty subset of $\mathbb{N}$ that does not have a least element. Define the proposition $P(n)$:="n is not an element of S" and then use complete induction to prove the result.

Problem 2.24 (Well-Ordering Principle)**.** Prove that every nonempty subset of the natural numbers has a least element.

It turns out that the Well-Ordering Principle and the Axiom of Induction are equivalent. In other words, one can prove the Well-Ordering Principle from the Axiom of Induction, as we have done, but one can also prove the Axiom of Induction if the Well-Ordering Principle is assumed.

The final result of this section can be thought of as a generalized version of the Well-Ordering Principle.

Problem 2.25. Prove that if A is a nonempty subset of the integers and there exists $b \in A^c$ such that $b \geq a$ for all $a \in A$, then A contains a greatest element.

In the previous problem, b is referred to as an upper bound for A. We will study upper bounds in Section 3.4.

> Nothing that's worth anything is ever easy.
>
> Mike Hall, ultra-distance cyclist

2.3 Functions

Let A and B be sets. A **relation** R **from** A **to** B is a subset of $A \times B$. If R is a relation from A to B and $(a,b) \in R$, then we say that a **is related to** b and we may write aRb in place of $(a,b) \in R$.

A function is a special type of relation, where the basic building blocks are a first set and a second set, say X and Y, and a "correspondence" that assigns *every* element of X to *exactly one* element of Y. More formally, if X and Y are nonempty sets, a **function** f **from** X **to** Y is a relation from X to Y such that for every $x \in X$, there exists a unique $y \in Y$ such that $(x,y) \in f$. The set X is called the **domain** of f and is denoted by $\boxed{\mathrm{Dom}(f)}$. The set Y is called the **codomain** of f and is denoted by $\boxed{\mathrm{Codom}(f)}$ while the subset of the codomain defined via

$$\boxed{\mathrm{Rng}(f) := \{y \in Y \mid \text{there exists } x \text{ such that } (x,y) \in f\}}$$

is called the **range** of f or the **image of** X under f.

There is a variety of notation and terminology associated to functions. We will write $\boxed{f : X \to Y}$ to indicate that f is a function from X to Y. We will make use of statements such as "Let $f : X \to Y$ be the function defined via…" or "Define $f : X \to Y$ via…", where f is understood to be a function in the second statement. Sometimes the word **mapping** (or **map**) is used in place of the word function. If $(a,b) \in f$ for a function f, we often write $\boxed{f(a) = b}$ and say that "f maps a to b" or "f of a equals b". In this case, a may be called an **input** of f and is the **preimage** of b under f while b is called an **output** of f and is the **image** of a under f. Note that the domain of a function is the set of inputs while the range is the set of outputs for the function.

According to our definition, if $f : X \to Y$ is a function, then every element of the domain is utilized exactly once. However, there are no restrictions on whether an element of the codomain ever appears in the second coordinate of an ordered pair in the relation. Yet if an element of Y is in the range of f, it may appear in more than one ordered pair in the relation.

It follows immediately from the definition of function that two functions are equal if and only if they have the same domain, same codomain, and the same set of ordered pairs in the relation. That is, functions f and g are equal if and only if $\mathrm{Dom}(f) = \mathrm{Dom}(g)$, $\mathrm{Codom}(f) = \mathrm{Codom}(g)$, and $f(x) = g(x)$ for all $x \in X$.

Since functions are special types of relations, we can represent them using digraphs and graphs when practical. Digraphs for functions are often called **function** (or **mapping**) **diagrams**. When drawing function diagrams, it is standard practice to put the vertices for the domain on the left and the vertices for the codomain on the right, so that all directed edges point from left to right. We may also draw an additional arrow labeled by the name of the function from the domain to the codomain.

Example 2.26. Let $X = \{a,b,c,d\}$ to $Y = \{1,2,3,4\}$ and define the relation f from X to Y via

$$f = \{(a,2),(b,4),(c,4),(d,1)\}.$$

Since each element X appears exactly once as a first coordinate, f is a function with domain X and codomain Y (i.e., $f : X \to Y$). In this case, we see that $\mathrm{Rng}(f) = \{1,2,4\}$. Moreover, we can write things like $f(a) = 2$ and $c \mapsto 4$, and say things like "f maps b to 4" and "the image of d is 1." The function diagram for f is depicted in Figure 2.1.

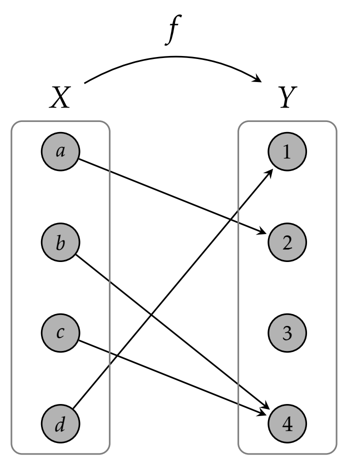

Figure 2.1: Function diagram for a function from $X = \{a,b,c,d,\}$ to $Y = \{1,2,3,4\}$.

Problem 2.27. What properties does the digraph for a relation from X to Y need to have in order for it to represent a function?

Problem 2.28. In high school I am sure that you were told that a graph represents a function if it passes the **vertical line test**. Carefully state what the vertical line test says and then explain why it works.

Sometimes we can define a function using a formula. For example, we can write $f(x) = x^2 - 1$ to mean that each x in the domain of f maps to $x^2 - 1$ in the codomain. However, notice that providing only a formula is ambiguous! A function is determined by its domain, codomain, and the correspondence between these two sets. If we only provide a description for the correspondence, it is not clear what the domain and codomain are. Two functions that are defined by the same formula, but have different domains or codomains are *not* equal.

Example 2.29. The function $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x^2 - 1$ is not equal to the function $g : \mathbb{N} \to \mathbb{R}$ defined by $g(x) = x^2 - 1$ since the two functions do not have the same domain.

Sometimes we rely on context to interpret the domain and codomain. For example, in a calculus class, when we describe a function in terms of a formula, we are implicitly assuming that the domain is the largest allowable subset of $\mathbb{R}$—sometimes called the **default domain**—that makes sense for the given formula while the codomain is $\mathbb{R}$.

Example 2.30. If we write $f(x) = x^2 - 1$, $g(x) = \sqrt{x}$, and $h(x) = \frac{1}{x}$ without mentioning the domains, we would typically interpret these as the functions $f : \mathbb{R} \to \mathbb{R}$, $g : [0, \infty) \to \mathbb{R}$, and $h : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ that are determined by their respective formulas.

Problem 2.31. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function f from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(f) = \mathrm{Codom}(f)$.

(b) A function g from a set with 4 elements to a set with 3 elements such that $\mathrm{Rng}(g)$ is strictly smaller than $\mathrm{Codom}(g)$.

There are a few special functions that we should know the names of. Let X and Y be nonempty sets.

- If $X \subseteq Y$, then the function $\iota : X \to Y$ defined via $\iota(x) = x$ is called the **inclusion map from X into Y**. Note that "ι" is the Greek letter "iota".
- If the domain and codomain are equal, the inclusion map has a special name. If X is a nonempty set, then the function $i_X : X \to X$ defined via $i_X(x) = x$ is called the **identity map** (or **identity function**) **on X**.
- Any function $f : X \to Y$ defined via $f(x) = c$ for a fixed $c \in Y$ is called a **constant function**.
- A **piecewise-defined function** (or **piecewise function**) is a function defined by specifying its output on a partition of the domain. Note that "piecewise" is a way of expressing the function, rather than a property of the function itself.

Example 2.32. The function $f : \mathbb{N} \to \mathbb{N}$ defined via

$$f(x) = \begin{cases} x^2 - 1, & \text{if } x \text{ is even,} \\ 17, & \text{if } x \text{ is odd} \end{cases}$$

is an example of a piecewise-defined function.

It is important to point out that not every function can be described using a formula! Despite your prior experience, functions that can be represented succinctly using a formula are rare.

The next problem illustrates that some care must be taken when attempting to define a function.

Problem 2.33. For each of the following, explain why the given description does not define a function.

(a) Define $f : \{1,2,3\} \to \{1,2,3\}$ via $f(a) = a - 1$.

(b) Define $g : \mathbb{N} \to \mathbb{Q}$ via $g(n) = \frac{n}{n-1}$.

(c) Let $A_1 = \{1,2,3\}$ and $A_2 = \{3,4,5\}$. Define $h : A_1 \cup A_2 \to \{1,2\}$ via

$$h(x) = \begin{cases} 1, & \text{if } x \in A_1 \\ 2, & \text{if } x \in A_2. \end{cases}$$

(d) Define $s : \mathbb{Q} \to \mathbb{Z}$ via $s(a/b) = a + b$.

In mathematics, we say that an expression is **well defined** (or **unambiguous**) if its definition yields a unique interpretation. Otherwise, we say that the expression is not well defined (or is **ambiguous**). For example, if $a, b, c \in \mathbb{R}$, then the expression abc is well defined since it does not matter if we interpret this as $(ab)c$ or $a(bc)$ since the real numbers are associative under multiplication.

When we attempt to define a function, it may not be clear without doing some work that our definition really does yield a function. If there is some potential ambiguity in the definition of a function that ends up not causing any issues, we say that the function is well defined. However, this phrase is a bit of misnomer since all functions are well defined. The issue of whether a description for a proposed function is well defined often arises when defining things in terms of representatives of equivalence classes, or more generally in terms of how an element of the domain is written. For example, the descriptions given in parts (c) and (d) of Problem 2.33 are not well defined. To show that a potentially ambiguous description for a function $f : X \to Y$ is well defined prove that if a and b are two representations for the same element in X, then $f(a) = f(b)$.

Let $f : X \to Y$ be a function.

- The function f is said to be **injective** (or **one-to-one**) if for all $y \in \text{Rng}(f)$, there is a unique $x \in X$ such that $y = f(x)$.
- The function f is said to be **surjective** (or **onto**) if for all $y \in Y$, there exists $x \in X$ such that $y = f(x)$.
- If f is both injective and surjective, we say that f is **bijective**.

An injective function is also called an **injection**, a surjective function is called a **surjection**, and a bijective function is called a **bijection**. To prove that a function $f : X \to Y$ is an injection, we must prove that if $f(x_1) = f(x_2)$, then $x_1 = x_2$. To show that f is surjective, you should start with an arbitrary $y \in Y$ and then work to show that there exists $x \in X$ such that $y = f(x)$.

Problem 2.34. Assume that X and Y are finite sets. Provide an example of each of the following. You may draw a function diagram, write down a list of ordered pairs, or write a formula as long as the domain and codomain are clear.

(a) A function $f : X \to Y$ that is injective but not surjective.

(b) A function $f : X \to Y$ that is surjective but not injective.

(c) A function $f : X \to Y$ that is a bijection.

(d) A function $f : X \to Y$ that is neither injective nor surjective.

Problem 2.35. Provide an example of each of the following. You may either draw a graph or write down a formula. Make sure you have the correct domain.

(a) A function $f : \mathbb{R} \to \mathbb{R}$ that is injective but not surjective.

(b) A function $f : \mathbb{R} \to \mathbb{R}$ that is surjective but not injective.

(c) A function $f : \mathbb{R} \to \mathbb{R}$ that is a bijection.

(d) A function $f : \mathbb{R} \to \mathbb{R}$ that is neither injective nor surjective.

(e) A function $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ that is injective.

Problem 2.36. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at most once*.

This statement is often called the **horizontal line test**. Explain why the horizontal line test is true.

Problem 2.37. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *at least once*.

Explain why this statement is true.

Problem 2.38. Suppose $X \subseteq \mathbb{R}$ and $f : X \to \mathbb{R}$ is a function. Fill in the blank with the appropriate word.

> The function $f : X \to \mathbb{R}$ is _________ if and only if every horizontal line hits the graph of f *exactly once*.

Explain why this statement is true.

Problem 2.39. Determine whether each of the following functions is injective, surjective, both, or neither. In each case, you should provide a proof or a counterexample as appropriate.

(a) Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$

(b) Define $g : \mathbb{R} \to [0, \infty)$ via $g(x) = x^2$

(c) Define $h : \mathbb{R} \to \mathbb{R}$ via $h(x) = x^3$

(d) Define $k : \mathbb{R} \to \mathbb{R}$ via $k(x) = x^3 - x$

(e) Define $c : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ via $c(x, y) = x^2 + y^2$

(f) Define $f : \mathbb{N} \to \mathbb{N} \times \mathbb{N}$ via $f(n) = (n, n)$

(g) Define $g : \mathbb{Z} \to \mathbb{Z}$ via

$$g(n) = \begin{cases} \frac{n}{2}, & \text{if } n \text{ is even} \\ \frac{n+1}{2}, & \text{if } n \text{ is odd} \end{cases}$$

(h) Define $\ell : \mathbb{Z} \to \mathbb{N}$ via

$$\ell(n) = \begin{cases} 2n + 1, & \text{if } n \geq 0 \\ -2n, & \text{if } n < 0 \end{cases}$$

The next two results should not come as as surprise.

Problem 2.40. Prove that the inclusion map $\iota : X \to Y$ for $X \subseteq Y$ is an injection.

Problem 2.41. Prove that the identity function $i_X : X \to X$ is a bijection.

If $f : X \to Y$ and $g : Y \to Z$ are functions, we define $g \circ f : X \to Z$ via $\boxed{(g \circ f)(x) = g(f(x))}$. The function $g \circ f$ is called the **composition of** f **and** g. It is important to notice that the function on the right is the one that "goes first." Moreover, we cannot compose any two random functions since the codomain of the first function must agree with the domain of the second function. In particular, $f \circ g$ may not be a sensible function even when $g \circ f$ exists. Figure 2.2 provides a visual representation of function composition in terms of function diagrams.

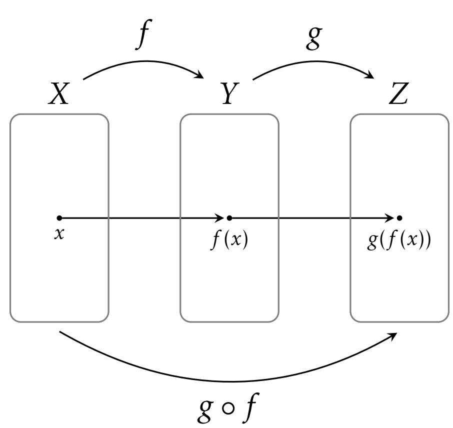

Figure 2.2: Visual representation of function composition.

Example 2.42. Consider the inclusion map $\iota : X \to Y$ such that X is a proper subset of Y and suppose $f : Y \to Z$ is a function. Then the composite function $f \circ \iota : X \to Z$ is given by

$$f \circ \iota(x) = f(\iota(x)) = f(x)$$

for all $x \in X$. Notice that $f \circ \iota$ is simply the function f but with a smaller domain. In this case, we say that $f \circ \iota$ is the **restriction of f to X**, which is often denoted by $\boxed{f|_X}$.

The next problem illustrates that $f \circ g$ and $g \circ f$ need not be equal even when both composite functions exist.

Problem 2.43. Define $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$ and $g(x) = 3x - 5$, respectively. Determine formulas for the composite functions $f \circ g$ and $g \circ f$.

The next problem tells us that function composition is associative.

Problem 2.44. Prove that if $f : X \to Y$, $g : Y \to Z$, and $h : Z \to W$ are functions, then $(h \circ g) \circ f = h \circ (g \circ f)$.

Problem 2.45. In each case, give examples of finite sets X, Y, and Z, and functions $f : X \to Y$ and $g : Y \to Z$ that satisfy the given conditions. Drawing a function diagram is sufficient.

(a) f is surjective, but $g \circ f$ is not surjective.

(b) g is surjective, but $g \circ f$ is not surjective.

(c) f is injective, but $g \circ f$ is not injective.

(d) g is injective, but $g \circ f$ is not injective.

Problem 2.46. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both surjective functions, then $g \circ f$ is also surjective.

Problem 2.47. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both injective functions, then $g \circ f$ is also injective.

Problem 2.48. Prove that if $f : X \to Y$ and $g : Y \to Z$ are both bijections, then $g \circ f$ is also a bijection.

Problem 2.49. Assume that $f : X \to Y$ and $g : Y \to Z$ are both functions. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $g \circ f$ is injective, then f is injective.

(b) If $g \circ f$ is injective, then g is injective.

(c) If $g \circ f$ is surjective, then f is surjective.

(d) If $g \circ f$ is surjective, then g is surjective.

There are two important types of sets related to functions. Let $f : X \to Y$ be a function.

- If $S \subseteq X$, the **image** of S under f is defined via

$$\boxed{f(S) := \{f(x) \mid x \in S\}}.$$

- If $T \subseteq Y$, the **preimage** (or **inverse image**) of T under f is defined via

$$\boxed{f^{-1}(T) := \{x \in X \mid f(x) \in T\}}.$$

The image of a subset S of the domain is simply the subset of the codomain we obtain by mapping the elements of S. It is important to emphasize that the function f maps *elements* of X to *elements* of Y, but we can apply f to a *subset* of X to yield a *subset* of Y. That is, if $S \subseteq X$, then $f(S) \subseteq Y$. Note that the image of the domain is the same as the range of the function. That is, $f(X) = \mathrm{Rng}(f)$.

When it comes to preimages, the notation $f^{-1}(T)$ should not be confused with an inverse function (which may or may not exist for an arbitrary function f). For $T \subseteq Y$, $f^{-1}(T)$ is the set of elements in the domain that map to elements in T. As a special case, $f^{-1}(\{y\})$ is the set of elements in the domain that map to $y \in Y$. If $y \notin \mathrm{Rng}(f)$, then $f^{-1}(\{y\}) = \emptyset$. Notice that if $y \in Y$, $f^{-1}(\{y\})$ is always a sensible thing to write while $f^{-1}(y)$ only makes sense if f^{-1} is a function. Also, note that the preimage of the codomain is the domain. That is, $f^{-1}(Y) = X$.

Problem 2.50. Define $f : \mathbb{Z} \to \mathbb{Z}$ via $f(x) = x^2$. List elements in each of the following sets.

(a) $f(\{0,1,2\})$

(b) $f^{-1}(\{0,1,4\})$

Problem 2.51. Find functions f and g and sets S and T such that $f(f^{-1}(T)) \neq T$ and $g^{-1}(g(S)) \neq S$.

Problem 2.52. Suppose $f : X \to Y$ is an injection and A and B are disjoint subsets of X. Are $f(A)$ and $f(B)$ necessarily disjoint subsets of Y? If so, prove it. Otherwise, provide a counterexample.

Problem 2.53. Let $f : X \to Y$ be a function and suppose $A, B \subseteq X$ and $C, D \subseteq Y$. Determine whether each of the following statements is true or false. If a statement is true, prove it. Otherwise, provide a counterexample.

(a) If $A \subseteq B$, then $f(A) \subseteq f(B)$.

(b) If $C \subseteq D$, then $f^{-1}(C) \subseteq f^{-1}(D)$.

(c) $f(A \cup B) \subseteq f(A) \cup f(B)$.

(d) $f(A \cup B) \supseteq f(A) \cup f(B)$.

(e) $f(A \cap B) \subseteq f(A) \cap f(B)$.

(f) $f(A \cap B) \supseteq f(A) \cap f(B)$.

(g) $f^{-1}(C \cup D) \subseteq f^{-1}(C) \cup f^{-1}(D)$.

(h) $f^{-1}(C \cup D) \supseteq f^{-1}(C) \cup f^{-1}(D)$.

(i) $f^{-1}(C \cap D) \subseteq f^{-1}(C) \cap f^{-1}(D)$.

(j) $f^{-1}(C \cap D) \supseteq f^{-1}(C) \cap f^{-1}(D)$.

(k) $A \subseteq f^{-1}(f(A))$.

(l) $A \supseteq f^{-1}(f(A))$.

(m) $f(f^{-1}(C)) \subseteq C$.

(n) $f(f^{-1}(C)) \supseteq C$.

We can generalize the results above to handle arbitrary collections of sets.

Problem 2.54. Let $f : X \to Y$ be a function and suppose $\{A_\alpha\}_{\alpha\in\Delta}$ is a collection of subsets of X. Prove each of the following.

(a) $f\left(\bigcup_{\alpha\in\Delta} A_\alpha\right) = \bigcup_{\alpha\in\Delta} f(A_\alpha)$.

(b) $f\left(\bigcap_{\alpha\in\Delta} A_\alpha\right) \subseteq \bigcap_{\alpha\in\Delta} f(A_\alpha)$.

Problem 2.55. Let $f : X \to Y$ be a function and suppose $\{C_\alpha\}_{\alpha\in\Delta}$ is a collection of subsets of Y. Prove each of the following.

(a) $f^{-1}\left(\bigcup_{\alpha\in\Delta} C_\alpha\right) = \bigcup_{\alpha\in\Delta} f^{-1}(C_\alpha)$.

(b) $f^{-1}\left(\bigcap_{\alpha\in\Delta} C_\alpha\right) = \bigcap_{\alpha\in\Delta} f^{-1}(C_\alpha)$.

In mathematics the art of proposing a question must be held of higher value than solving it.

Georg Cantor, mathematician