Chapter 5

Continuity

Definition 5.1. We say that a function f is *continuous at a point* x in its domain (or at the point $(x, f(x))$) if, for any open interval S containing $f(x)$, there is an open interval T containing x such that if $t \in T$ is in the domain of f, then $f(t) \in S$.

Definition 5.2. A function f is *continuous* if it is continuous at every point in its domain.

Let's show that this definition of continuity behaves the way we expect from calculus.

Exercise 5.3. Show that each of the following functions is continuous.

(a) $f : \mathbb{R} \to \mathbb{R}$ defined via $f(x) = x$.

(b) $g : \mathbb{R} \to \mathbb{R}$ defined via $g(x) = 2x$.

(c) $h : \mathbb{R} \to \mathbb{R}$ defined via $h(x) = x + 3$.

Problem 5.4. Show that any linear function given by $f(x) = mx + b$ is continuous for all $x \in \mathbb{R}$.

The next problem tells us that we can reframe continuity in terms of distance.

Problem 5.5. Let f be a function. Prove that f is continuous at x if and only if for every $\epsilon > 0$, then there exists $\delta > 0$ so that if t is in the domain of f and $|t - x| < \delta$, then $|f(t) - f(x)| < \epsilon$.

The previous characterization is typically referred to as the "$\epsilon - \delta$ definition of continuity", although for us it is a theorem instead of a definition. This characterization is used as the definition of continuity in metric spaces.

Problem 5.6. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in [0,1] \\ 0, & \text{otherwise.} \end{cases}$$

Find all points x where f is continuous and justify your answer.

Problem 5.7. Define $g : \{0\} \to \mathbb{R}$ via $g(0) = 0$. Show that g is continuous at $x = 0$.

Problem 5.8. Define $f : \mathbb{R} \to \mathbb{R}$ via

$$f(x) = \begin{cases} 1, & \text{if } x \in \mathbb{Q} \\ 0, & \text{otherwise.} \end{cases}$$

Find all points x where f is continuous and justify your answer.

Problem 5.9. Define $f : \mathbb{R} \to \mathbb{R}$ via $f(x) = x^2$. Prove that f is continuous.

Exercise 5.10. Find a continuous function f and an open interval U such that the preimage $f^{-1}(U)$ is not an open interval.

Problem 5.11. Let f be a function. Prove that f is continuous if and only if the preimage $f^{-1}(U)$ of every open set U is an open set intersected with the domain of f.

The previous characterization of continuity is often referred to as the "open set definition of continuity" and is the definition used in topology.

It turns out that there is a deep connection between continuity and sequences!

Definition 5.12. We say that a function f is *sequentially continuous at a point* x if, for every sequence $(x_i)_{i=1}^{\infty}$ (in the domain of f) converging to x, it is also true that $(f(x_i))_{i=1}^{\infty}$ converges to $f(x)$.

Problem 5.13. Let f be a function. Prove that f is continuous at x if and only if f is sequentially continuous at x.

The upshot of the previous problem is that the notions of being *continuous at a point* and *sequentially continuous at a point* are equivalent on the real numbers. However, there are contexts in mathematics where the two are not equivalent. This is topic in a branch of mathematics called *topology*. If you want to know more, check out the following YouTube video:

https://www.youtube.com/watch?v=sZ5fBHGYurg

The sequential way of thinking of continuity often makes proving some basic facts concerning continuity easier.

At this point, we have four different ways of thinking about continuity.

- Definition 5.1 using open intervals.
- Problem 5.5 using ϵ and δ.
- Problem 5.11 using inverse images of open sets.
- Problem 5.13 using sequential continuity.

You should take the time to review each one. Moreover, it is worth pointing out that three of the four characterizations involve continuity at a point. Which one does not? For the remainder of the book, feel free to use which ever characterization you'd like.

Problem 5.14. Suppose f and g are functions that are continuous at x and let $c \in \mathbb{R}$. Prove that each of the following functions are also continuous at x.

(a) cf

(b) $f + g$

(c) $f - g$

(d) fg

Exercise 5.15. Prove that every polynomial is continuous on all of $\mathbb{R}$.

Problem 5.16. Suppose $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and consider the closed interval $[a,b]$. Is the image $f([a,b])$ always a closed interval? If so, prove it. Otherwise, provide a counterexample.

Problem 5.17. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and K is a compact point set, then the image $f(K)$ is compact.

The next result tells us that continuous functions always attain a maximum value on closed intervals. Of course, we have an analogous result involving minimums.

Problem 5.18 (Extreme Value Theorem)**.** Let I be a closed interval. Prove that if f is continuous on I, then there exists $x_M \in I$ such that $f(x_M) \geq f(x)$ for all $x \in I$.

Problem 5.19. Is the hypothesis that I is closed needed in the Extreme Value Theorem? Justify your answer.

Problem 5.20. Is the converse of the Extreme Value Theorem true? That is, if a function attains a maximum value over a closed interval, does that imply that the function is continuous. If so, prove it. Otherwise, provide a counterexample.

Problem 5.21. Let $f : [0,1] \to \mathbb{R}$ and assume that the image $f([0,1])$ has a supremum. Show there is a sequence of points $(x_i)_{i=1}^{\infty}$ in $[0,1]$ such that $(f(x_i))_{i=1}^{\infty}$ converges to that supremum. Does this show that f is continuous on $[0,1]$?

Definition 5.22. We say that a point set M is *disconnected* if there exists two disjoint open sets U_1 and U_2 such that $M \cap U_1$ and $M \cap U_2$ are nonempty but $M \subseteq U_1 \cup U_2$ (equivalently, $M = (M \cap U_1) \cup (M \cap U_2)$). If a point set is non disconnected, then we say that it is *connected*.

Exercise 5.23. Provide examples of sets that are disconnected. Also, provide some examples of sets that are connected. In each case, try to find examples with various other properties such as open, closed, neither open nor closed, bounded, unbounded, and compact. You do not need to worry about justifying your examples in this exercise.

Problem 5.24. Determine whether each of the following point sets is connected or disconnected. Prove your answers.

(a) $\mathbb{Q}$

(b) $[0,1] \cup [2,3]$

(c) $\{\frac{1}{n} \mid n \in \mathbb{N}\}$

(d) $\{17\}$

(e) $\emptyset$

The next problem is harder than it looks.

Problem 5.25. Prove that every closed interval $[a,b]$ is connected.

The next problem is analogous to Problem 5.17. It also likely captures your intuition about continuity from high school and calculus.

Problem 5.26. Prove that if $f : \mathbb{R} \to \mathbb{R}$ is a continuous function and M is a connected point set, then the image $f(M)$ is connected.

Problem 5.27 (Intermediate Value Theorem)**.** Let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < 0 < f(b)$ or $f(a) > 0 > f(b)$, then there exists $r \in I$ such that $f(r) = 0$.

We can generalize the previous result, which is also often referred to as the Intermediate Value Theorem.

Problem 5.28. Let $I = [a,b]$ be a closed interval. Prove that if f is continuous on I such that $f(a) < c < f(b)$ or $f(a) > c > f(b)$, then there exists $r \in I$ such that $f(r) = c$.

Problem 5.29. Is the hypothesis that I is closed needed in the Intermediate Value Theorem? Justify your answer.

Problem 5.30. Is the converse of the Intermediate Value Theorem true? If so, prove it. Otherwise, provide a counterexample.

Problem 5.31. Let $f : [0,1] \to \mathbb{R}$ be a function such that $f(0) = -1$, $f(1) = 1$, and $f([0,1]) = \{-1,1\}$. Prove that there exists $x \in [0,1]$ such that f is not continuous at x.