Homework 12

Combinatorics

This assignment is optional. If you choose to complete this assignment, your score on this assignment will replace your lowest score on a previous assignment, possibly one you did not complete.

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, Discord, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. A *walk* of length ℓ in a graph G is a sequence of vertices $W : v_0, v_1, \ldots, v_\ell$ such that $v_{i-1}v_i \in E$ for all $1 \leq i \leq \ell$. We say that the walk is from v_0 to v_ℓ. We call W a *path* if all the vertices are distinct. We call W a *cycle* if $v_0 = v_\ell$, all other vertices are distinct, and all edges are distinct. For a digraph (directed graph) G, we define *directed walk*, *directed path*, and *directed cycle* in the obvious ways. Prove that if G is a digraph, then any directed walk of length at least 2 from u to v with $u = v$ contains a directed cycle.

2. Determine the chromatic number for each of the following families of graphs. For this problem, you do *not* need to justify your answer.
   (a) K_n (complete graph with n vertices)
   (b) P_n (path graph with n vertices)
   (c) C_n (cycle graph with n vertices)

3. Determine the chromatic polynomial for each of the following families of graphs. For this problem, you do *not* need to justify your answer.
   (a) $G = (V, E)$, where $|V| = n$ and $E = \emptyset$ (i.e., G consists of n vertices and no edges).
   (b) K_n
   (c) P_n

4. A *tree* is a connected graph that does not contain any cycles. It is easy to prove using induction that a tree with n vertices has $n-1$ edges. You can take this for granted. Prove that the chromatic polynomial for any tree G with n vertices is $P(G;t) = t(t-1)^{n-1}$. Consider using induction.

5. Let G be a graph and write

$$P(G;t) = a_0 t^n - a_1 t^{n-1} + a_2 t^{n-2} - \cdots + (-1)^n a_n t^0,$$

where $a_0 \neq 0$. Define $\text{mdeg}(P(G;t))$ to be the largest $n-k$ such that $a_k \neq 0$ (i.e., smallest exponent such that corresponding coefficient is nonzero). Prove each of the following. Consider using induction and the Deletion-Contraction Lemma.

(a) $|V| = n$.

(b) $\text{mdeg}(P(G;t))$ equals the number of connected components of G.

(c) $a_k \geq 0$ for all $0 \leq k \leq n$ and $a_k > 0$ for all $0 \leq k \leq n - \text{mdeg}(P(G;t))$.

(d) $a_0 = 1$ and $a_1 = |E|$.

6. An *orientation* O of a graph G is a digraph with the same vertex set obtained by replacing each edge $\{u,v\}$ of G by one of the possible directed edges (u,v) or (v,u). An orientation is called *acyclic* if it does not contain any directed cycles. Let $\mathcal{A}(G)$ denote the collection of acyclic orientations of G and let $a(G) := |\mathcal{A}(G)|$. Determine each of the following. For this problem, you do *not* need to justify your answer.

   (a) $a(P_n)$

   (b) $a(C_n)$

7. Prove that $a(K_n) \geq n!$. It turns out that $a(K_n) = n!$. If you can prove this stronger result, go for it. For this problem, avoid using any theorems involving $P(K_n;t)$.