Exam 2 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 2 is worth a total of 26 points and is worth 30% of your overall score on Exam 2. This take-home exam is due by 5PM **Wednesday, April 10**. Your overall score on Exam 2 is worth 20% of your overall grade. Good luck and have fun!

I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems or problems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem X or Problem Y, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. In this problem, we will explore an alternate approach o Problem 98. Suppose we have n labeled balls, say $1,2,\ldots,n$ that we place in k distinct boxes. Each such arrangement corresponds to a *barred permutation*, where we draw a vertical bar for the divisions between the boxes (so that the number of boxes is one more than the number of bars) and by way of standardization, we list the balls in each box in increasing order. For example, | |56|2| |14| | |3 is a barred permutation that indicates that we have nine boxes such that balls 5 and 6 were placed in the third box, ball 2 in the fourth box, balls 1 and 4 in the sixth box, ball 3 in the ninth box, and all other boxes are empty.
   (a) (2 points) Explain why the number of ways to place n labeled balls in k distinct boxes is k^n.
   (b) (2 points) If we fix $n \geq 1$, explain why the generating function for the number of ways to place n labeled balls in distinct boxes is given by
   $$\sum_{k\geq 0} k^n t^k.$$
   (c) (2 points) We can partition the set of all barred permutations (arrangements of balls in boxes) according to the underlying permutations. Since we require numbers to increase within a box, we know that there must be a bar in each descent position of a barred permutation, but otherwise, we can insert bars into the gaps between the numbers, to the left of the leftmost number, or to the right of the rightmost number at will. Argue that the generating function corresponding to the identity permutation $w = 12\cdots n$ is
   $$\frac{t}{(1-t)^{n+1}}.$$
   Hint: Recall that $\frac{1}{1-t} = 1 + t + t^2 + \cdots$. Also, we have $n+1$ places to place or not place bars.
   (d) (2 points) Argue that all the generating function for an arbitrary $w \in S_n$ is given by
   $$\frac{t^{\text{runs}(w)}}{(1-t)^{n+1}}.$$
   (e) (2 points) Conclude that
   $$\sum_{k\geq 0} k^n t^k = \sum_{w\in S_n} \frac{t^{\text{runs}(w)}}{(1-t)^{n+1}}.$$

   Recall that the nth Eulerian polynomial is given by $A_n(t) = \sum_{w\in S_n} t^{\text{runs}(w)}$. Thus, we have
   $$\sum_{k\geq 0} k^n t^k = \frac{A_n(t)}{(1-t)^{n+1}},$$
   which yields the result in Problem 98.

2. (4 points each) There are over two hundred different sets of combinatorial objects that are enumerated by Catalan numbers. It is great fun to find bijections between these sets, and to try to count them in a manner that gives Narayana numbers. In fact, some of my previous undergraduate research students discovered what I believe to be a previously unknown collection of objects that are counted by the Catalan numbers (and we used generating functions to prove our result). Each of the following is enumerated by the Catalan numbers (the sequence may be shifted). Here some possible methods of attack for verifying this claim:

- Show that the collection is in bijection with a collection that is known to be counted by the Catalan numbers.
- Show that the collection satisfies the initial condition and recurrence of the Catalan numbers.
- Show that the collection is enumerated by the closed form formula that yields the Catalan numbers.
- Show that the collection has the same generating functions as the Catalan numbers.

For **four** of the following, verify that the collection is enumerated by the Catalan numbers. Also, if you wish, you may rely upon the fact that an earlier collection in the list is enumerated by the Catalan numbers even if you did not prove it.

(a) (123-avoiding permutations) A permutation is 123-avoiding if there is no triple of indices $i < j < k$ such that $w(i) < w(j) < w(k)$. The set of 123-avoiders is denoted $S_n(123)$.

(b) (Noncrossing partitions) A noncrossing partition is a set partition of $\{1, 2, \dots, n\}$ such that no two of its blocks, say A and B, contain members $a, c \in A$ and $b, d \in B$ such that $a < b < c < d$. Here are the noncrossing partitions on three elements: $\{\{1\},\{2\},\{3\}\}, \{\{1,2\},\{3\}\}, \{\{1,3\},\{2\}\}, \{\{1\},\{2,3\}\}$, and $\{\{1,2,3\}\}$. These are all set partitions of $\{1,2,3\}$. For $n = 4$, there are 15 set partitions, and they are all noncrossing except for $\{\{1,3\},\{2,4\}\}$.

(c) (Balanced parenthesizations) A sequence of parentheses is balanced if it can be parsed syntactically. In other words, there should be the same number of open parentheses "(" and closed parentheses ")", and when reading from left to right there should never be more closed parentheses than open. Here are the five balanced parenthesizations containing three pairs: ()()(), ()(()), (()()), (())(), ((())).

(d) (Two-row standard Young tableaux) A standard Young tableau is a two dimensional array of numbers (from 1 to the number of entries in the array) that increases across rows and down columns. Let $SYT(2,n)$ denote the number standard Young tableaux in a $2 \times n$ rectangular array. There is a figure containing the standard Young tableau for $n = 3$ on page 119.

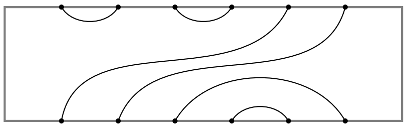
(e) (Triangulations of a polygon) See Figure 9.3 on page 120. Here we dissect an $(n+2)$-gon with n triangles. Notice the polygon is fixed in space, so one might as well label the vertices. (Incidentally, this is the problem that Euler was interested in when he studied the Catalan numbers!)

(f) (Decreasing binary trees) See Figure 9.4 on page 120. Here n is the number of vertices of degree two. Notice that how these are drawn in the plane matters, i.e., left and right subtrees matter.

(g) (Temperley–Lieb n-diagrams) Consider a rectangle with n nodes across the top edge and n nodes across the bottom edge. A n-diagram in the rectangle consists of n edges connecting the $2n$ nodes such that:

  - each edge connects a pair of nodes,
  - each node is connected to exactly one edge,
  - the edges must be drawn inside the rectangle,
  - none of the edges are allowed to cross or touch.

  Below is an example of a 6-diagram.



Let TL(n) denote the collection of n-diagrams.

3. **Bonus Question!** (2 points) This question is optional. For **two** of the collections you chose in the previous problem, describe where the Narayana numbers make an appearance. You only need to briefly justify your answers.