Final Exam

Your Name:

Names of Any Collaborators:

Instructions

Submit your solutions to the following questions by noon on **Friday, May 6**.

This exam is worth a total of 26 points and is worth 25% of your overall grade. I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts. Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing the following problems, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. Further information:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Problem 3.16, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.** To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. (2 points each) For $\boxed{\textbf{three}}$ of the following, provide an example with the stated properties. In each case, *briefly* justify your answer.

   (a) Provide an example of a signed permutation that is a fortress.

   (b) Provide an example of signed permutation in $\pi \in S_{11}^{\pm}$ such that $d_r(\pi) = 9$.

   (c) Provide an example of a signed permutation $\pi \in S_{10}^{\pm}$ such that the shift class of π consists of 11 signed permutations.

   (d) Provide an example of a signed permutation $\pi \in S_{10}^{\pm}$ such that $\mathrm{cyc}(\pi) = 1$.

   (e) Provide an example of a signed permutation $\pi \in S_{11}^{\pm}$ such that $\mathrm{cyc}(\pi) = 2$.

   (f) Provide an example of a signed permutation $\pi \in S_4^{\pm}$ such that $\pi \circ \rho \leq 4$ for every reversal ρ. *Note:* Since the maximal reversal distance in $S_4^{\pm}$ is 5, the existence of such an example says that there are permutations that are maximal is the poset sense that are not maximal in terms of reversal distance.

   (g) Provide an example of two signed permutation $\pi, \sigma \in S_4^{\pm}$ such that $d_r(\pi) = d_r(\sigma)$ and there exists a reversal ρ such that $\pi = \sigma \circ \rho$.

2. (4 points each) Complete $\boxed{\textbf{four}}$ of the following.

   (a) How many signed permutations in $S_n^{\pm}$ have the property that no single reversal applied to π will eliminate a breakpoint? Justify your answer.

   (b) Prove that there exists an optimal sorting sequence in terms of reversals for every $\pi \in S_n^{\pm}$ that never breaks an adjacency.

   (c) Prove that there is a bijection from S_n to the collection of signed permutations with no negative signs in $S_n^{\pm}$ that preserves the number of cycles in the respective breakpoint diagrams.

   (d) Suppose n is odd with $n \geq 5$. Prove that there are no signed permutations in $S_n^{\pm}$ whose breakpoint diagrams consist of a single cycle and do not consist of any negative signs.

   (e) Prove that if $\pi \in S_n^{\pm}$ is a fortress, then $d_r(\pi) \leq n - 1$.

   (f) Suppose $\pi \in S_n^{\pm}$. Prove that π includes at least one negative number if and only if $\mathrm{BG}(\pi)$ contains an oriented cycle.

   (g) Suppose $n \geq 4$ and let $\pi \in S_n^{\pm}$ be maximal. Prove that either every cycle of $\mathrm{BG}(\pi)$ is a minimal hurdle or there exists a unique cycle that is a maximal hurdle and all other cycles are minimal hurdles. Moreover, prove that if $\mathrm{BG}(\pi)$ has a maximal hurdle, then there exists $\gamma \in [\pi]$, where $\mathrm{BG}(\gamma)$ does not have a maximal hurdle.

   (h) Use the following result to prove that the number of maximal permutations in $S_6^{\pm}$ is 180.

3. (4 points) Recall that $\mathrm{rk}_{\max}(S_n^{\pm}, d_r)$ denotes the number of permutations in $S_n^{\pm}$ that attain the maximal reversal distance (i.e., the number of maximal permutations). Prove that for $n \neq 1, 3$, we have

$$\mathrm{rk}_{\max}(S_n^{\pm}, d_r) = \sum_{(\alpha_1,\ldots,\alpha_k) \in C_{\mathrm{odd}}^{>1}(n+1)} \left(\prod_{i=1}^{k} \frac{2(\alpha_i - 1)!}{\alpha_i + 1} \right) \cdot \begin{cases} \alpha_1, & \text{if } k \neq 1 \\ 1, & \text{if } k = 1. \end{cases}$$

