Homework 5

Combinatorics of Genome Rearrangements

You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in his or her own work. In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments with no questions asked**.

Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. However, when it comes to completing assignments for this course, you should *not* look to resources outside the context of this course for help. That is, you should not be consulting the web, other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. This includes Chegg and Course Hero. On the other hand, you may use each other, the textbook, me, and your own intuition. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.** Please read NAU's Academic Integrity Policy.

Complete the following problems.

1. Prove that as posets, the interval below the n-cycle $(1,2,\dots,n)$ in the sorting graph for S_n using the collection of transpositions as a generating set is isomorphic to $\mathrm{NC}(n)$. That is, prove $\mathrm{SG}([e,(1,2,\dots,n)],R_t) \cong \mathrm{NC}(n)$.

2. Verify that the proposed bijection $\phi : S_n(231) \to \mathrm{NC}(n)$ that was presented in class is in fact a well-defined bijection that maps maximal decreasing runs to blocks of a noncrossing partition.

3. A **Dyck path** of length $2n$ is a lattice path from $(0,0)$ to (n,n) consisting of n horizontal steps "East" from (i,j) to $(i+1,j)$ and n vertical steps "North" from (i,j) to $(i,j+1)$ such that all points on the path satisfy $i \le j$ (i.e., when drawn in the Cartesian plane, the path lies on or above the line $y=x$). The set of all Dyck paths of length $2n$ is denoted $\mathrm{Dyck}(n)$. For this homework, let $d_n := |\mathrm{Dyck}(n)|$.

   (a) Prove that $d_n = C_n$ by describing a bijection between $\mathrm{Dyck}(n)$ and $S_n(231)$. There are a few different ways to attack this problem, but one way is to think of 231-avoiding permutations as non-attacking rook arrangements (ask me if you don't know what this means). See if you can find a natural correspondence to a Dyck path for a given non-attacking rook arrangement for a 231-avoiding permutation.

   (b) Prove that there is a bijection from the set of lattice paths from $(0,0)$ to (n,n) that pass below $y=x$ to the set of lattice paths from $(0,0)$ to $(n+1,n-1)$. *Hint:* Consider the first point on lattice path from $(0,0)$ to (n,n) that passes below $y=x$. Reflect the remaining portion of the path over the appropriate line to get a path from $(0,0)$ to $(n+1,n-1)$.

   (c) Prove that $d_n = \binom{2n}{n} - \binom{2n}{n-1}$. *Hint:* The number of lattice paths (not just Dyck paths) from $(0,0)$ to (n,n) is equal to $\binom{2n}{n}$ while the number of lattice paths from $(0,0)$ to $(n+1,n-1)$ is equal to $\binom{2n}{n-1}$.

You just proved that $C_n = \binom{2n}{n} - \binom{2n}{n-1}$, and it is easy to verify that $\binom{2n}{n} - \binom{2n}{n-1} = \frac{1}{n+1}\binom{2n}{n}$, so that $C_n = \frac{1}{n+1}\binom{2n}{n}$, as well.

4. There are over two hundred different sets of combinatorial objects that are known to be enumerated by Catalan numbers. It is great fun to find bijections between these sets, and to try to count them in a manner that manifests the Narayana numbers. Here are some possible methods of attack for proving that a collection of objects is enumerated by the Catalan numbers:

   - Show that the collection is in bijection with a collection that is known to be counted by the Catalan numbers.
   - Show that the collection satisfies the initial condition and recurrence of the Catalan numbers.
   - Show that the collection is enumerated by the closed form formula that we found in Problem 3.
   - Show that the collection has the same generating functions as the Catalan numbers.

   For **one** of the following, verify that the collection is enumerated by the Catalan numbers. If you wish, you may rely upon the fact that an earlier collection in the list is enumerated by the Catalan numbers even if you did not prove it.

   (a) (Triangulations of a polygon) A **triangulation** of a convex $(n+2)$-gon is a dissection into n triangles using only lines from vertices to vertices. Think of the polygon as being fixed in space. Incidentally, this is the problem that Euler was interested in when he studied the Catalan numbers!

   (b) (Planar binary trees) A **planar binary tree** is a rooted tree such that every internal node has precisely two children. Since these are rooted trees, how they are drawn in the plane matters, i.e., left and right subtrees matter. If there are n internal nodes, then there are $n+1$ leaves. Let $\mathrm{PB}(n)$ denote the collection of planar binary trees with n internal nodes.

   (c) (Two-row standard Young tableaux) A **standard Young tableau** is a two dimensional array of numbers (from 1 to the number of entries in the array) that increases across rows and down columns. Let $\mathrm{SYT}(2,n)$ denote the collection of standard Young tableaux in a $2 \times n$ rectangular array.

5. A **stack-sortable permutation** is a permutation whose elements may be sorted by an algorithm whose internal storage is limited to a single stack data structure. More precisely, consider $\pi = \pi_1\pi_2\cdots\pi_n \in S_n$ and consider three stacks arranged from left to right. Our starting configuration is an empty left stack, $\pi_1\pi_2\cdots\pi_n$ in the middle where we think of π_1 being at the top of the stack and π_n at the bottom of the stack, and an empty right stack. We then employ the following algorithm:

   (1) Move the current top number π_i of the middle stack to the top of the left stack if the left stack is empty or if π_i is smaller than the current top value π_j of the left stack.

   (2) Otherwise, move the top value π_j from the left stack to the bottom of the right stack.

Let $\mathcal{S}(\pi_1\pi_2\cdots\pi_n)$ denote the final arrangement of the numbers in the right stack. We say that $\pi_1\pi_2\cdots\pi_n$ is *stack sortable* if $\mathcal{S}(\pi_1\pi_2\cdots\pi_n) = 12\cdots n$.

(a) If $\pi_i = n$, prove that

$$\mathcal{S}(\pi_1\cdots\pi_{i-1}n\pi_{i+1}\cdots\pi_n) = \mathcal{S}(\pi_1\cdots\pi_{i-1})\mathcal{S}(\pi_{i+1}\cdots\pi_n)n.$$

(b) Enumerate the number of stack-sortable permutations in S_n. *Hint:* Induction could come in handy depending on your approach.