Homework 1

Combinatorial Game Theory

Let's begin with a few reminders. Homework will consist of a mixture of the following:

- Problems that are modifications of examples we have discussed in class.
- Problems that extend concepts introduced in class.
- Problems that introduce new concepts not yet discussed in class.
- Problems that synthesize multiple concepts that we either introduced in class or in a previous homework problem.

Some homework problems will be straightforward while others are intended to be challenging. You should anticipate not knowing what to do on some of the problems at first glance. You may have several false starts. Some frustration, maybe even a lot of frustration, should be expected. This is part of the natural learning process. On the other hand, it is not my intention to leave you to fend for yourselves. I am here to help and I want to help. You are encouraged to seek assistance from your classmates (while adhering to the **Rules of the Game**) and from me. Please visit office hours and ask questions on our Q&A Discussion board. I am always willing to give hints/nudges, so please ask.

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Consider Cram(B), where B is an $m \times n$ grid (with $m, n \geq 1$).
   (a) If m and n are even, which player has a winning strategy? Explain your answer.
   (b) If m is even and n is odd, which player has a winning strategy? Explain your answer.

2. The game CHOMP($m \times n$) is played on an $m \times n$ grid with the square in the lower-left corner designated as poisoned. On a player's turn, they select an available non-poisoned square, which removes the selected square and all squares above and to the right. Assume normal play (i.e., the player that removes all remaining non-poisoned squares is the winner). An alternate interpretation is that the player that is forced to choose the poisoned square is the loser. Explain why the first player always has a winning strategy when played on a grid that is larger than 1×1. *Hint:* What happens when the first player chooses the upper-right square? Consider the two possibilities. For this problem, it is more likely that you will find a non-constructive proof that does not give explicit information about what the winning strategy actually is.

3. Determine which SUBTRACTION(n;1,3,4) are P-positions. Justify your answer.

4. Suppose the positions of a finite impartial game can be partitioned into mutually exclusive sets A and B with the following properties:

   - Every option of a position in A is in B; and
   - Every position in B has at least one option in A.

   Prove that A is the set of P-positions and B is the set of N-positions. *Hint:* Use induction.

5. Prove that every position of a finite rulegraph R with a unique source is a subposition of the starting position. A consequence is that a finite rulegraph with a unique source is automatically a gamegraph. We do not need to check that every position is a subposition of the starting position.

6. Find an example to illustrate that the previous result is not true without the finiteness assumption.

7. Recall that a **multiset** M is an unordered collection of elements that may be repeated. To distinguish between sets and multisets, we will use the notation $\{\}$ versus $\{\!\{\}\!\}$. For example, $M = \{\!\{a,a,a,b,c,c\}\!\}$ is a multiset (not to be confused with the set containing the set $\{a,b,c\}$; mathematical notation is both awesome and frustrating at times). Grundy's game played on a multiset M, denoted GRUNDY(M), is a game in which two players alternate choosing an element a from the current multiset and replacing it with two nonzero elements b and c such that $a = b + c$ and $b \neq c$.

   (a) Draw the gamegraph for GRUNDY($\{\!\{7\}\!\}$). Label each position as a P-position or N-position.

   (b) Which player has a winning strategy for GRUNDY($\{\!\{7\}\!\}$)? Explain how one can utilize the gamegraph together with the P and N labels to determine a winning strategy for this player.