Homework 3

Combinatorial Game Theory

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Let o^+ denote the outcome function for normal play. Prove that if G is a gamegraph, then $o^+(\mathsf{G}+\mathsf{G}) = P$. That is, $\mathsf{G}+\mathsf{G}$ is a P-position (i.e., the second player has a winning strategy).

2. Provide examples of gamegraphs G and H that satisfy each of the following. If such an example does not exist, explain why.

   (a) $o^+(\mathsf{G}) = P$ and $o^+(\mathsf{H}) = P$ while $o^+(\mathsf{G}+\mathsf{H}) = N$

   (b) $o^+(\mathsf{G}) = P$ and $o^+(\mathsf{H}) = P$ while $o^+(\mathsf{G}+\mathsf{H}) = P$

   (c) $o^+(\mathsf{G}) = P$ and $o^+(\mathsf{H}) = N$ while $o^+(\mathsf{G}+\mathsf{H}) = N$

   (d) $o^+(\mathsf{G}) = P$ and $o^+(\mathsf{H}) = N$ while $o^+(\mathsf{G}+\mathsf{H}) = P$

   Note: I really should have included (e) $o^+(\mathsf{G}) = N$ and $o^+(\mathsf{H}) = N$ while $o^+(\mathsf{G}+\mathsf{H}) = N$, and (f) $o^+(\mathsf{G}) = N$ and $o^+(\mathsf{H}) = N$ while $o^+(\mathsf{G}+\mathsf{H}) = P$. If you are bored, ponder those as well. But if you don't, it will not count against you.

3. Let $\alpha : \mathsf{G} \to \mathsf{H}$ be an option preserving gamegraph map. Prove that α is source preserving if and only if α is surjective.

4. Provide an example that illustrates that the previous problem does not generalize to rulegraphs. That is, find an option preserving rulegraph map that takes sources to sources injectively but is not surjective.

5. Let $\alpha : C \to D$ be an option preserving digraph map.

   (a) Prove $q \in \mathrm{Opt}(p)$ in C implies $\alpha(q) \in \mathrm{Opt}(\alpha(p))$ in D.

(b) Prove that if $\alpha(q) \in \mathrm{Opt}(\alpha(p))$, then $\alpha(q) = \alpha(r)$ for some r satisfying $r \in \mathrm{Opt}(p)$.

Note: Taken together, these results tell us that every option preserving rulegraph map is a faithful digraph homomorphism. In this context, faithful means that image of map is an induced subgraph.

6. Prove that the inverse of a bijective option preserving rulegraph map $\alpha : \mathsf{R} \to \mathsf{S}$ is also option preserving.