Homework 4

Combinatorial Game Theory

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Consider the option preserving gamegraph map α from Problem 4 on Homework 2.
   (a) Identify the equivalence classes induced by the kernel of α.
   (b) Let $\sim$ denote the kernel of α. Draw the quotient gamegraph $(\textsc{Nim}(2)+\textsc{Nim}(3))/\sim$, where the positions are labeled using the equivalence classes from part (a).
2. Consider $\textsc{Cram}(B)$, where B is an $n\times n$ grid. For positions p and q, define $p\sim q$ if we can obtain q from p by applying a rotation or reflection symmetry to B. Let's take for granted that $\sim$ is a congruence relation. If B is a 3×3 grid, draw the quotient gamegraph for $\textsc{Cram}(B)/\sim$. Label the formal birthday of each position. Assuming normal play, also label each position as a P-position or N-position.
3. A map $f : V(\mathsf{R}) \to Y$ on the positions of a rulegraph R is called a **valuation** if there is a map $\mu : 2^Y \to Y$ such that $f(p) = \mu(f(\mathrm{Opt}(p)))$ for each position p of R (where 2^Y is the power set of a set Y). Note that fixing a μ determines a corresponding f that is defined for any rulegraph. So we are going to denote this valuation by f even if we are working with several rulegraphs. If f is a valuation on a gamegraph G with starting position s, then we write $f(\mathsf{G}) := f(s)$.
   (a) Verify that the normal play outcome function $o^+ : V(\mathsf{R}) \to \{P,N\}$ is a valuation by identifying the appropriate μ.
   (b) Verify that the misere play outcome function $o^- : V(\mathsf{R}) \to \{P,N\}$ is a valuation by identifying the appropriate μ.
   (c) Verify that the formal birthday $\mathrm{fbd}(p)$ of a position p is a valuation by identifying the appropriate μ.

(d) Verify that the minimum distance of a position from a terminal position is a valuation by identifying the appropriate μ.

4. Prove that if $\alpha : \mathsf{R} \to \mathsf{S}$ is an option preserving rulegraph map and f is a valuation, then $f(\alpha(p)) = f(p)$ for each position p in R. *Hint:* Use fancy induction.

5. Suppose $\alpha : \mathsf{G} \to \mathsf{H}$ is a source and option preserving gamegraph map. Explain why $o^+(\mathsf{G}) = o^+(\mathsf{H})$, $o^-(\mathsf{G}) = o^-(\mathsf{H})$, and $\text{fbd}(\mathsf{G}) = \text{fbd}(\mathsf{H})$.

6. Prove that if $\sim$ is a congruence relation on a rulegraph R and q is a subposition of p, then p and q are not related. In particular, if R is a gamegraph, then the congruence class of the starting position s is $[s] = \{s\}$. Loosely speaking, this result tells us that there is no "vertical collapsing" in a quotient rulegraph.