Homework 6

Combinatorial Game Theory

Please review the **Rules of the Game** from the syllabus. Reviewing material from previous courses and looking up definitions and theorems you may have forgotten is fair game. Since mathematical reasoning, problem solving, and critical thinking skills are part of the learning outcomes of this course, all assignments should be prepared by the student. Developing strong competencies in this area will prepare you to be a lifelong learner and give you an edge in a competitive workplace. When it comes to completing assignments for this course, unless explicitly told otherwise, you should *not* look to resources outside the context of this course for help. That is, you should *not* be consulting the web (e.g., Chegg and Course Hero), generative artificial intelligence tools (e.g., ChatGPT), mathematics assistive technologies (e.g., Wolfram Alpha and Photomath), other texts, other faculty, or students outside of our course in an attempt to find solutions to the problems you are assigned. On the other hand, you may use each other, the textbook, me, and your own intuition. You are highly encouraged to seek out assistance by asking questions on Discord. You are allowed and encouraged to work together on homework. Yet, each student is expected to turn in their own work. **If you feel you need additional resources, please come talk to me and we will come up with an appropriate plan of action.**

In general, late homework will not be accepted. However, you are allowed to turn in **up to two late homework assignments**. Unless you have made arrangements in advance with me, homework turned in after class will be considered late.

Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers.

1. Given a gamegraph G, if there exists position p with $\text{fbd}(p) = 1$ (so that G has at least one edge), define $\hat{\mathsf{G}}$ to be the gamegraph obtained from G by deleting all terminal positions (and arrows that point at terminal positions). For any gamegraph G, define $\mathsf{G}^\bullet$ be the gamegraph obtained from G by including a new position $t^\bullet$ such that $\text{Opt}(t) = \{t^\bullet\}$ for all terminal positions t of G (so that $t^\bullet$ is the unique terminal position of $\mathsf{G}^\bullet$). Determine whether each of the following statements is true or false. If true, prove it. Otherwise, provide a counterexample.
   (a) $o^-(\mathsf{G}) = o^+(\hat{\mathsf{G}})$
   (b) $o^+(\mathsf{G}) = o^-(\mathsf{G}^\bullet)$
   (c) $\text{nim}(\mathsf{G}) = \text{nim}(\hat{\mathsf{G}})$
   (d) $\text{nim}(\mathsf{G}) = \text{nim}(\mathsf{G}^\bullet)$
   (e) $o^-(\mathsf{G} + \mathsf{X}) = o^+(\hat{\mathsf{G}} + \mathsf{X})$ for all gamegraphs X
   (f) $o^+(\mathsf{G} + \mathsf{X}) = o^-(\mathsf{G}^\bullet + \mathsf{X})$ for all gamegraphs X

We now introduce two games played on finite groups. Let G be a finite group. For both games, the starting position is the empty set. The players take turns choosing previously-unselected elements to create jointly-selected sets of elements, which are the positions of the game. In the avoidance game $\mathsf{DNG}(G)$, the first player chooses $x_1 \in G$ such that $\langle x_1 \rangle \neq G$ and at the kth turn, the designated player selects $x_k \in G \setminus \{x_1, \ldots, x_{k-1}\}$ such that $\langle x_1, \ldots, x_k \rangle \neq G$ to create the position $\{x_1, \ldots, x_k\}$. Notice that the positions of $\mathsf{DNG}(G)$ are exactly the non-generating subsets of G. A position Q is an option of P if $Q = P \cup \{g\}$ for some $g \in G \setminus P$, where $\langle Q \rangle \neq G$. The player who cannot select an element without building a generating set is the loser. We note that there is no avoidance game for the trivial group since the empty set generates the whole group.

In the achievement game $\mathsf{GEN}(G)$, the first player chooses any $x_1 \in G$ and at the kth turn, the designated player selects $x_k \in G \setminus \{x_1, \ldots, x_{k-1}\}$ to create the position $\{x_1, \ldots, x_k\}$. A player wins on the nth turn if $\langle x_1, \ldots, x_n \rangle = G$. In this case, the positions of $\mathsf{GEN}(G)$ are subsets of terminal positions, which are certain generating sets of G. Note that the second player has a winning strategy if G is trivial since $\langle \emptyset \rangle = G$, so the first player has no legal opening move.

It turns out that both games fit the framework that Nandor introduced, where $\mathsf{DNG}(G)$ and $\mathsf{GEN}(G)$ correspond to $\mathsf{AVD}(H)$ and $\mathsf{ACV}(H)$, respectively, where $H = (G, \mathcal{H})$ is a hypergraph with the set of hyperedges $\mathcal{H}$ being equal to the family of minimal generating sets of G. In $\mathsf{GEN}(G)$, the game ends as soon as a position contains a hyperedge. In $\mathsf{DNG}(G)$, positions may never contain a hyperedge. In this context, the set $\mathcal{S}_H$ of maximal stable sets is equal to the collection of maximal subgroups.

2. Consider $G = \mathbb{Z}_2 \times \mathbb{Z}_2 = \{(0,0), (1,0), (0,1), (1,1)\}$ (under operation of addition mod 2 in each component).
   (a) Draw the gamegraph for $\mathsf{DNG}(G)$. Label each position with the corresponding nim-value. Which player has a winning strategy?
   (b) Draw the minimum quotient gamegraph for $\mathsf{DNG}(G)$. Label each position with the corresponding nim-value.
   (c) Draw the gamegraph for $\mathsf{GEN}(G)$. Label each position with the corresponding nim-value. Which player has a winning strategy?
   (d) Draw the minimum quotient gamegraph for $\mathsf{GEN}(G)$. Label each position with the corresponding nim-value.
   (e) Identify the set of hyperedges $\mathcal{H}$ that corresponds to $\mathsf{GEN}(G)$.
   (f) Identify the set $\mathrm{Tr}(\mathcal{H})$ of minimal transversals.
   (g) Determine whether $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H})) = \mathcal{H}$ by doing the computation on the left.
   (h) Identify the set $\mathcal{S}_H$ of maximal stable sets.
   (i) Verify that $\mathcal{S}_H = \complement(\mathrm{Tr}(\mathcal{H}))$ by doing the computation on the right.
   (j) Draw the structure diagram for $\mathsf{GEN}(G)$.

3. Consider $G = \mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$ (under operation of addition mod 6).
   (a) Draw the gamegraph for $\mathsf{DNG}(G)$. Label each position with the corresponding nim-value. Which player has a winning strategy?
   (b) Draw the minimum quotient gamegraph for $\mathsf{DNG}(G)$. Label each position with the corresponding nim-value.
   (c) Draw the gamegraph for $\mathsf{GEN}(G)$. Label each position with the corresponding nim-value. Which player has a winning strategy?
   (d) Draw the minimum quotient gamegraph for $\mathsf{GEN}(G)$. Label each position with the corresponding nim-value.
   (e) Identify the set of hyperedges $\mathcal{H}$ that corresponds to $\mathsf{GEN}(G)$.
   (f) Identify the set $\mathrm{Tr}(\mathcal{H})$ of minimal transversals.
   (g) Determine whether $\mathrm{Tr}(\mathrm{Tr}(\mathcal{H})) = \mathcal{H}$ by doing the computation on the left.
   (h) Identify the set $\mathcal{S}_H$ of maximal stable sets.
   (i) Verify that $\mathcal{S}_H = \complement(\mathrm{Tr}(\mathcal{H}))$ by doing the computation on the right.
   (j) Draw the structure diagram for $\mathsf{GEN}(G)$.