Homework X

Discrete Mathematics

This assignment is optional. If you choose to complete it, your score will either replace a current missing assignment or replace your current lowest homework score (if the score on this assignment is higher than your lowest homework score).

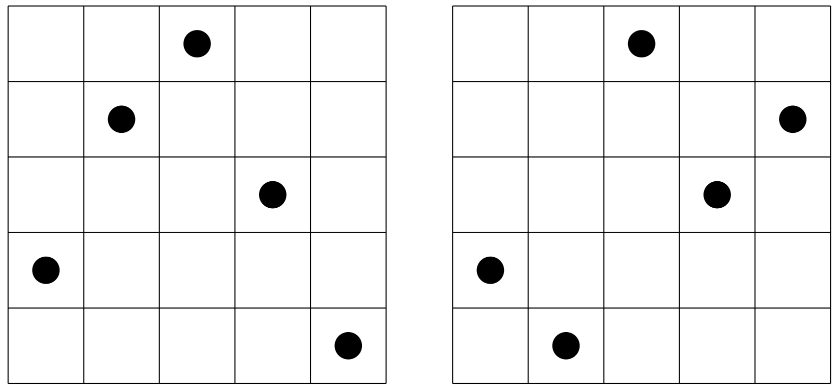
Complete the following problems. Unless explicitly stated otherwise, you are expected to justify your answers. In many problems this means that you should use words to describe what you are doing and why. In other problems, simply providing sufficient arithmetic may be sufficient. If a problem asks you to count something, please box your final answer.

1. If you have enough toothpicks, you can make a large triangular grid. The size 1 grid requires 3 toothpicks, the size 2 grid requires 9 toothpicks, etc. Let t_n be the number of toothpicks required to make a size n triangular grid, so that $t_1 = 3$, $t_2 = 9$, and so on.
   (a) Find a recursive definition for the sequence. Don't forget any necessary initial conditions.
   (b) Is the sequence arithmetic or geometric? If not, is it the sequence of partial sums of an arithmetic or geometric sequence?
   (c) Find a closed formula for the nth term of the sequence.

2. Given a second-order linear constant-coefficient homogeneous recurrence relation $a_n = c_1a_{n-1} + c_2a_{n-2}$, the solution described in Theorem 9.18 only works if there are two distinct roots of the corresponding equation. It turns out if there is a repeated characteristic root r (i.e., the characteristic equation can be factored as $0 = (x-r)^2$, so that r is the only characteristic root), then the solution to the recurrence relation is
$$a_n = ar^n + bnr^n,$$
   where a and b are constants determined by the initial conditions. Assuming this result, solve the recurrence relation $a_n = 6a_{n-1} - 9a_{n-2}$ with initial conditions $a_0 = 1$ and $a_1 = 4$.

3. Tickets to a show are 50 cents and $2n$ customers stand in a queue at the ticket window. Half of them have \$1 and the others have 50 cents. The cashier starts with no money. How many arrangements of the queue are possible with the proviso that the cashier always be able to make change? *Hint:* Find a bijection to Dyck paths...

4. We say that a non-attacking rook arrangement on an $n \times n$ chessboard is **medium-high-low-avoiding** if when scanning the heights of the rooks from left to right, for each rook a we never see both a rook b to the right (not necessarily consecutively) that is higher than a followed (not necessarily consecutively) by a rook c that is lower than both a and b. For example, consider the non-attacking rook arrangements given below. The arrangement on the left has a few occurrences of a medium-high-low pattern, namely involving columns 1, 2, 5; columns 1, 3, 5; columns 1, 4, 5; columns 2, 3, 4; columns 2, 3, 5. On the other hand, the arrangement on the right is medium-high-low-avoiding.



How many medium-high-low-avoiding non-attacking rook arrangements are there on an $n \times n$ chessboard? *Hint:* Verify initial conditions and recurrence for Catalan numbers.