Name:

Names of Any Collaborators:

Instructions

This portion of Exam 2 is worth a total of 20 points and is due at the beginning of class on **Monday, April 3**. Your total combined score on the in-class portion and take-home portion is worth 15% of your overall grade. I expect your solutions to be *well-written, neat, and organized*. Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any results that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem xyz, then you should say so.
2. Unless you prove them, you cannot use any results that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

1. For any $n \in \mathbb{N}$, say that n straight lines are "safely drawn in the plane" if no two of them are parallel and no three of them meet in a single point. Let $S(n)$ be the number of regions formed when n straight lines are safely drawn in the plane.
   (a) (2 points) Compute $S(1)$, $S(2)$, $S(3)$, and $S(4)$.
   (b) (2 points) Conjecture a recursive formula for $S(n)$; that is, a formula for $S(n)$ which may involve some of the previous terms $\{S(n-1), S(n-2), \ldots\}$. (If necessary, first compute a few more values of $S(n)$.)
   (c) (4 points) Prove your conjecture.

2. (4 points) Prove that every nonempty subset of the natural numbers contains a least element.*

3. (4 points) Prove **one** of the following theorems.

   Theorem C.1. Suppose we draw n lines safely in the plane (see Problem 1). This partitions the plane into disjoint regions (some of which are polygons with finite area and some are not). Suppose we color each of the regions so that no two adjacent regions (i.e., share a common edge) have the same color. Then we can color the regions using two colors.

   Theorem C.2. Every amount of postage that is at least 12 cents can be made from 4-cent and 5-cent stamps.

   Theorem C.3. A certain type of chessboard is 2^n squares wide and 2^n squares long and one of the squares has been cut out, but you don't know which one! You have a bunch of L-shapes made up of 3 squares. Prove that you can perfectly cover this chessboard with the L-shapes (with no overlap) for any $n \in \mathbb{N}$. Figure 5.1 in the course notes depicts one possible covering for the case involving $n = 2$.

4. (4 points) Prove **one** of the following theorems. For both theorems, you will need to make use of Definitions 6.23, 6.28, and 6.33 in Chapter 6 of our course notes.

   Theorem D.1. Suppose $\sim$ is an equivalence relation on a set A and let $a, b \in A$. Then $[a] = [b]$ iff $a \sim b$.†

   Theorem D.2. Suppose $\sim$ is an equivalence relation on a set A. Then
   (a) $\bigcup_{x \in A} [x] = A$, and
   (b) for all $x, y \in A$, either $[x] = [y]$ or $[x] \cap [y] = \emptyset$.‡

*This theorem is known as the Well-Ordering Principle (WOP). *Hint:* Towards a contradiction, suppose S is a nonempty subset of $\mathbb{N}$ that does not have a least element. Define the proposition $P(n)$:="n is not an element of S". Use induction.

†This is Theorem 6.39 in the course notes.

‡This is Theorem 6.40 in the course notes.