MAT 411: Introduction to Abstract Algebra
Exam 3 (Take-Home Portion)

Your Name:

Names of Any Collaborators:

Instructions

This portion of Exam 3 is worth a total of 32 points and is due at the beginning of class on **Wednesday, April 27**. Your total combined score on the in-class portion and take-home portion is worth 15% of your overall grade.

I expect your solutions to be *well-written, neat, and organized.* Do not turn in rough drafts. What you turn in should be the "polished" version of potentially several drafts.

Feel free to type up your final version. The LaTeX source file of this exam is also available if you are interested in typing up your solutions using LaTeX. I'll gladly help you do this if you'd like.

The simple rules for the exam are:

1. You may freely use any theorems that we have discussed in class, but you should make it clear where you are using a previous result and which result you are using. For example, if a sentence in your proof follows from Theorem 1.41, then you should say so.
2. Unless you prove them, you cannot use any results from the course notes that we have not yet covered.
3. You are **NOT** allowed to consult external sources when working on the exam. This includes people outside of the class, other textbooks, and online resources.
4. You are **NOT** allowed to copy someone else's work.
5. You are **NOT** allowed to let someone else copy your work.
6. You are allowed to discuss the problems with each other and critique each other's work.

I will vigorously pursue anyone suspected of breaking these rules.

You should **turn in this cover page** and all of the work that you have decided to submit. **Please write your solutions and proofs on your own paper.**

To convince me that you have read and understand the instructions, sign in the box below.

Signature:

Good luck and have fun!

Problem 1. (2 points each) Consider the alternating group A_4. Lagrange's Theorem tells us that the possible orders of subgroups for A_4 are 1, 2, 3, 4, 6, and 12.

(a) Write down all of the elements of order 2 in A_4.

(b) Argue that any subgroup of A_4 that contains any two elements of order 2 must contain a subgroup isomorphic to V_4.

(c) Argue that if A_4 has a subgroup of order 6, that it cannot be isomorphic to R_6.

(d) It turns out that up to isomorphism, there are only two groups of order 6, namely S_3 and R_6. Suppose that H is a subgroup of A_4 of order 6. Part (c) guarantees that $H \cong S_3$. Argue that H must contain all of the elements of order 2 from A_4.

(e) Explain why A_4 cannot have a subgroup of order 6.

The next few problems explore the concept of quotient groups. For additional information, check out Section 8.2 of the course notes.

Suppose G is an arbitrary group and let $H \leq G$. Consider the set of left cosets of H and define

$$(aH)(bH) := (ab)H.$$

The natural question to ask is whether this operation is well-defined. That is, does the result of multiplying two left cosets depend on our choice of representatives? More specifically, suppose $c \in aH$ and $d \in bH$. Then $cH = aH$ and $dH = bH$. According to the operation defined above, $(cH)(dH) = cdH$. It better be the case that $cdH = abH$, otherwise the operation is not well-defined.

Problem 2. (4 points) Let G be a group and let $H \leq G$. Prove that left coset multiplication (as defined above) is well-defined if and only if $H \trianglelefteq G$. *Note:* This is Theorem 8.29.

Problem 3. (4 points) Let G be a group and let $H \trianglelefteq G$. Prove that the set of left cosets of H in G forms a group under left coset multiplication. *Note:* This is Theorem 8.30.

The group from Problem 3 is denoted by G/H, read "G mod H", and is referred to as the **quotient group** of G by H. If G is a finite group, then G/H is exactly the group that arises from "gluing together" the colored blocks in a checkerboard-patterned group table. It's also the group that we get after applying the quotient process to the Cayley diagram. It's important to point out once more that this only works properly if H is a normal subgroup.

If $H \trianglelefteq G$, then $|G/H|$ is the number of left cosets of H in G. That is, $|G/H| = [G : H]$. In particular, if G is finite, then $|G/H| = |G|/|H|$.

Problem 4. (4 points) Let G be a group. Prove that if $G/Z(G)$ is cyclic, then G is abelian.*

Problem 5. (4 points each) Prove any **two** of the following.

(a) Let G be a group and let $H \trianglelefteq G$. Prove that if G is cyclic, then so is G/H. *Note:* This is Theorem 8.39.

(b) Prove that if $|G| = pq$, where p and q are primes (not necessarily distinct), then either $Z(G) = \{e\}$ or G is abelian.

(c) Suppose G is a group of order pq such that p and q are distinct primes (i.e, $p \neq q$). Prove that G must have both an element of order p and an element of order q.

*Note that since the elements of $Z(G)$ commute with all the elements of G, the left and right cosets of $Z(G)$ will be equal, and hence $Z(G)$ is normal in G.

(d) Let H be a normal subgroup of a group G. Prove that if $g \in G$, then the order of gH (in G/H) divides the order of g (in G).

Problem 6. (2 points) Show that the converse of Problem 5(a) is not true by providing a specific counterexample. *Note:* You should do this problem regardless of whether you chose to prove Problem 5(a).