Exam 1 (In-Class Portion)

1. There are lots of examples. The smallest such example is $V_4 \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Other examples include $S_3 \cong D_3$ and Q_8.

2. Assume $\varphi: Q_8 \longrightarrow \mathbb{Z}_2 \times \mathbb{Z}_2$ is a group homomorphism s.t. $\varphi(i) = (1,0)$ and $\varphi(j) = (0,1)$. Recall that a homomorphism is uniquely determined by its action on a generated set. By looking at the Cayley diagram for Q_8, it is clear that $Q_8 = \langle i, j \rangle$. In particular:

$$1 = i^4$$

$$-1 = i^2$$

$$k = ij$$

$$-k = ji$$

$$-i = i^3$$

$$-j = j^3$$

(a) Using the information above, we see that

$$\varphi(1) = (0,0)$$

$$\varphi(-1) = \varphi(i^2)$$
$$= \varphi(i) + \varphi(i)$$
$$= (1,0) + (1,0)$$
$$= (0,0)$$

$$\varphi(k) = \varphi(ij)$$
$$= \varphi(i) + \varphi(j)$$
$$= (1,0) + (0,1)$$
$$= (1,1)$$

$$\varphi(-k) = \varphi(-1 \cdot k)$$
$$= \varphi(-1) + \varphi(k)$$
$$= (0,0) + (1,1)$$
$$= (1,1)$$

$$\varphi(-i) = \varphi(-1 \cdot i)$$
$$= \varphi(-1) + \varphi(i)$$
$$= (0,0) + (1,0)$$
$$= (1,0)$$

$$\varphi(-j) = \varphi(-1 \cdot j)$$
$$= \varphi(-1) + \varphi(j)$$
$$= (0,0) + (0,1)$$
$$= (0,1).$$

Therefore, $\ker(\varphi) = \{\pm 1\}$.

(b) The work above makes it clear that $\varphi^{-1}((1,1)) = \{\pm k\}$.

3. Recall that

$$C_{D_8}(\langle s \rangle) = \{g \in D_8 \mid gx = xg \ \forall x \in \langle s \rangle\}.$$

Since $\langle s \rangle = \{e, s\}$ and e commutes with every element in D_8, we only need to find the elements in D_8 that commute w/ s. Clearly, $\{e, s\} \subseteq C_{D_8}(\langle s \rangle)$. Also, since $C_{D_8}(\langle s \rangle)$ is a subgroup of D_8, the possible orders of $C_{D_8}(\langle s \rangle)$ are: 2, 4, 8 (by Lagrange's Theorem).

Since $sr^2 = r^2 s$, $r^2 \in C_{D_8}(\langle s \rangle)$. This implies that $sr^2 \in C_{D_8}(\langle s \rangle)$ by closure (since both s, r^2 are in $C_{D_8}(\langle s \rangle)$. Now, the possible orders are 4 and 8.

However, $sr \neq rs$, and hence $|C_{D_8}(\langle s \rangle)| \neq 8$.

Therefore, $C_{D_8}(\langle s \rangle) = \{e, s, r^2, sr^2\}$

(which happens to be isomorphic to V_4).

4. (a) This one is "no." That is, $\mathrm{Two}(G)$ is not always a subgrp of G. For example, Consider $G = D_8$. Then $\mathrm{Two}(G) = \{e, s, sr, sr^2, sr^3, r^2\}$. This set is not closed, and so it is not a subgrp. Another thing to notice is that it contains 5 elmts, which violates Lagrange's Thm since 5 does not divide 8.

(b) This one is "yes."

Pf: Suppose $\varphi: G_1 \to G_2$ is a group hom and let $H_2 \leq G_2$. Define $H_1 = \varphi^{-1}(H_2)$. Clearly, $H_1 \neq \emptyset$. Let $g_1, h_1 \in H_1$. Then $\exists\, g_2, h_2 \in H_2$ s.t. $\varphi(g_1) = g_2$ and $\varphi(h_1) = h_2$. Since H_2 is a group, $g_2 h_2 \in H_2$. This implies that $\varphi(g_1 h_1) = \varphi(g_1)\varphi(h_1) = g_2 h_2 \in H_2$. But then $g_1 h_1 \in \varphi^{-1}(H_2) = H_1$. So, H_1 is closed. It remains to check inverses. Let $h_1 \in H_1$. Then $\exists$ $h_2 \in H_2$ s.t. $\varphi(h_1) = h_2$. Since H_2 is a group, $h_2^{-1} \in H_2$. This implies that $\varphi(h_1^{-1}) = \varphi(h_1)^{-1} = h_2^{-1} \in H_2$, and so $h_1^{-1} \in \varphi^{-1}(H_2) = H_1$. Therefore, $H_1 \leq G_1$. $\blacksquare$

(c) This one is "yes."

Pf: Let G be a grp and let $a, b \in G$. For sake of a contradiction, assume $|ab| \neq |ba|$. Wlog, suppose $|ab| < |ba|$. Say, $|ab| = k$ and $|ba| = n$. We see that

$$e = \underbrace{(ba)(ba)\cdots(ba)}_{n \text{ pairs}}$$

$$= b\underbrace{(ab)\cdots(ab)}_{k \text{ pairs}}a\underbrace{(ba)\cdots(ba)}_{n-k-1 \text{ pairs}}$$

$$= ba\underbrace{(ba)\cdots(ba)}_{n-k-1 \text{ pairs}}$$

since $(ab)^k = e$. But this contradicts $|ba| = n$. $\blacksquare$

5. Recall that a homomorphism is uniquely determined by its action on a generating set. Since $\mathbb{Z}_{10}$ is cyclic, we only need to determine where we can send the singleton generators. By a thm from class, we know that $\mathbb{Z}_{10}$ is generated by the elmts in $\mathbb{Z}_{10}$ that are relatively prime to 10; namely 1, 3, 5, 7. In order to obtain ~~any~~ an iso from $\mathbb{Z}_{10}$ to $\mathbb{Z}_{10}$, we must map singleton generators to singleton generators. For simplicity, we can focus on where we send 1. There are 4 automorphisms:

$\varphi_1(1) = 1$ (identity map)

$\varphi_2(1) = 3$

$$\varphi_3(1) = 5$$

$$\varphi_4(1) = 7.$$

Again, the action on the remaining elmts is uniquely determined in each case. In summary,

$$\mathrm{Aut}(\mathbb{Z}_{10}) = \{\varphi_1, \varphi_2, \varphi_3, \varphi_4\}$$

(which happens to be isomorphic to $\mathbb{Z}_4$).

6. This one is my favorite!

(a) First, observe that

$$(14) = (12)(23)(34)(23)(12) = s_1 s_2 s_3 s_2 s_1.$$

Then

$$\begin{aligned}(14)\cdot 1 &= (s_1 s_2 s_3 s_2 s_1)\cdot 1 \\ &= s_1 s_2 s_3 s_2 \cdot 4 \\ &= s_1 s_2 s_3 \cdot 6 \\ &= s_1 s_2 \cdot 6 \\ &= s_1 \cdot 4 \\ &= 1.\end{aligned}$$

(b) By part (a), (14) is in the stabilizer of 1. Also, it's clear that $(1), S_2 = (23)$ are also in the stabilizer. Since the stabilizer is a subgrp, we must also have $(14)(23)$ in the stabilizer by closure. That's 4 elmts, so we're done. In summary, the stabilizer is

$$\{(1), (14), (23), (14)(23)\},$$

which happens to be iso to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

(c) By (b), we know the kernel is contained in $\{(1), (14), (23), (14)(23)\}$. The only elmts from this list that fixes 5 is (1). Thus, the kernel is trivial, and hence the action is faithful.

An upshot of the work we just did is that the group that rearranges 3 coins and flips over evenly many is iso to S_4 (which might be a bit surprising).

7. (a) Label the 4 corners of a tetrahedron w/ 1, 2, 3, 4. Clearly, the group of rigid motions of the tetrahedron determines an action on $\{1,2,3,4\}$. If G is the group of rigid motions, then we have a hom from $G \longrightarrow S_4$. It is clear that the action is faithful (since every rigid motion moves at least one vertex). This implies that the hom $G \longrightarrow S_4$ is an injection. Therefore, G is iso to a subgrp of S_4. By the way,

this group happens to have size 12 and is iso to the alternating grp A_{12}. $\blacksquare$

(b) S'pose $\varphi: G \longrightarrow H$ is a grp hom. Certainly, $N_G(\ker(\varphi)) \subseteq G$. It remains to show the reverse containment. Let $g \in G$ and $k \in \ker(\varphi)$. We see that

$$\begin{aligned}\varphi(gkg^{-1}) &= \varphi(g)\varphi(k)\varphi(g^{-1})\\ &= \varphi(g)\cdot\varphi(g)^{-1}\\ &= e_2,\end{aligned}$$

where e_2 is identity in H. This shows that $gkg^{-1} \in \ker(\varphi)$, and so $g \in N_G(\ker(\varphi))$. Thus, $G \subseteq N_G(\ker(\varphi))$.

$\blacksquare$

8. See class notes and previous HW.

9. (a) Let G be a grp and define

$\varphi : G \to G$ via $\varphi(g) = g^2 \; \forall g \in G$.

($\Rightarrow$) Assume φ is a hom. Let $g, h \in G$. Then on one hand,

$$\varphi(gh) = (gh)^2$$

and on the other

$$\varphi(gh) = \varphi(g)\varphi(h) = g^2h^2.$$

This implies that

$$(gh)^2 = g^2h^2$$

$$\cancel{g}hg\cancel{h} = \cancel{g}gh\cancel{h}$$

$$hg = gh,$$

and so G is abelian.

($\Leftarrow$) Now, assume G is abelian.

Let $g, h \in G$. We see that

$$\begin{aligned}
\varphi(gh) &= (gh)^2 \\
&= ghgh \\
&= gghh \quad \text{(since } G \text{ abelian)} \\
&= g^2h^2 \\
&= \varphi(g)\varphi(h).
\end{aligned}$$

Therefore, φ is a grp hom. $\blacksquare$

(b) Sorry ... this problem broken since I left out 3/4 of a sentence.