Your Name:

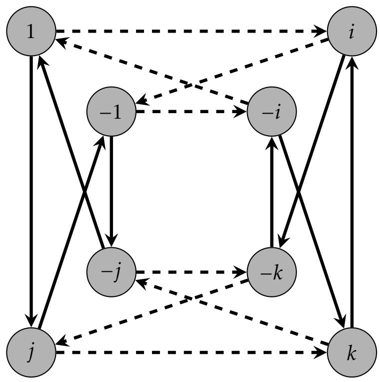
Instructions

Answer each of the following questions on your own paper. I expect your solutions to be *well-written, neat, and organized*. When you are finished, staple this sheet to the top of the work you wish me to grade. There are total number of 42 points up for grabs on this portion of the exam. The take-home portion of Exam 1 will be posted on the course webpage later today and will be due on Monday, October 19 at the beginning of class. Good luck and have fun!

Problems

1. (2 points) Provide an example of a group G such that all the proper subgroups of G are cyclic, but G is not cyclic.

2. (2 points each) Below is the Cayley diagram for the group Q_8 using the generating set $\{i, j\}$.



Suppose $\phi : Q_8 \to \mathbb{Z}_2 \times \mathbb{Z}_2$ is a group homomorphism such that $\phi(i) = (1,0)$ and $\phi(j) = (0,1)$.

   (a) Find $\ker(\phi)$.

   (b) Find $\phi^{-1}((1,1))$.

3. (2 points) Consider the group $D_8 = \langle r, s\rangle$. Find $C_{D_8}(\langle s\rangle)$. Justify your answer.

4. (4 points each) Complete **two** of the following. In each case, determine whether the answer to the question is 'yes' or 'no'. If the answer is 'yes', prove it. If 'no', provide a specific counterexample and briefly explain why it is a counterexample.

   (a) Let G be a group and define $\mathrm{Two}(G) = \{x \in G \mid x^2 = e\}$. Is $\mathrm{Two}(G)$ a subgroup of G?

   (b) Suppose $\phi : G_1 \to G_2$ is a group homomorphism and let $H_2 \leq G_2$. Define $H_1 = \phi^{-1}(H_2)$. Is H_1 a subgroup of G_1?

   (c) Let G be a group. Is it true that $|ab| = |ba|$ for all $a, b \in G$?

5. (4 points) Recall that $\mathrm{Aut}(G)$ is the group of automorphisms of a group G. Find $\mathrm{Aut}(\mathbb{Z}_{10})$. Briefly justify your answer.

6. (2 points each) Consider the group S_4. For brevity, write $s_1 = (1,2), s_2 = (2,3), s_3 = (3,4)$. Suppose S_4 acts on the set $\{1,2,3,4,5,6\}$ such that the following holds:

$s_1 \cdot 1 = 4$	$s_2 \cdot 1 = 1$	$s_3 \cdot 1 = 3$
$s_1 \cdot 2 = 3$	$s_2 \cdot 2 = 2$	$s_3 \cdot 2 = 4$
$s_1 \cdot 3 = 2$	$s_2 \cdot 3 = 5$	$s_3 \cdot 3 = 1$
$s_1 \cdot 4 = 1$	$s_2 \cdot 4 = 6$	$s_3 \cdot 4 = 2$
$s_1 \cdot 5 = 5$	$s_2 \cdot 5 = 3$	$s_3 \cdot 5 = 5$
$s_1 \cdot 6 = 6$	$s_2 \cdot 6 = 4$	$s_3 \cdot 6 = 6$

This looks way more complicated than it is. Here's a model to keep in mind. Imagine you have 3 cards sitting side-by-side such that the left card has a 1 on top and a 2 on the bottom, the middle card has 3 on top and a 4 on the bottom, and the right card has a 5 on top and a 6 on the bottom. Then s_1 simultaneously swaps and flips over the left and middle cards, s_2 swaps the middle and right cards (but doesn't flip them over), and s_3 swaps the left and middle cards (but doesn't flip them over). Assuming that what I've just described is an action of S_4 on $\{1,2,3,4,5,6\}$, determine each of the following.

   (a) Determine the action of $(1,4)$ on 1. That is, find $(1,4) \cdot 1$.

   (b) Find the stabilizer of 1. *Hint:* It has size 4 (you can assume this).

   (c) Is the action faithful? Briefly explain your answer.

7. (4 points) Prove **one** of the following.

   (a) Prove that the group of rigid motions of a tetrahedron is isomorphic to a subgroup of S_4.

   (b) Suppose $\phi : G \to H$ is a group homomorphism. We know that $\ker(\phi) \leq G$. Prove that $N_G(\ker(\phi)) = G$.

8. (4 points) Prove **two** of the following.

   (a) Let G be a group and let $x \in G$. Prove that $x^m = e$ iff $|x|$ divides m.

   (b) Suppose $\phi : G \to H$ is a group homomorphism. Prove that ϕ is an injection iff $\ker(\phi) = \{e\}$.

   (c) Suppose G is a group acting on the set A (assume action is on the left). For each $g \in G$, define $\sigma_g : A \to A$ via $\sigma_g(a) = g \cdot a$ for all $a \in A$. Assuming σ_g is a permutation, prove that the function $\phi : G \to S_A$ given by $\phi(g) = \sigma_g$ is a group homomorphism.

9. (4 points) Prove **one** of the following.

   (a) Let G be a group and define $\phi : G \to G$ via $\phi(g) = g^2$ for all $g \in G$. Prove that ϕ is a homomorphism iff G is abelian.

   (b) Define the group $G = \{z \in \mathbb{C} \mid |z| < \infty\}$ where the operation is ordinary multiplication of complex numbers.* Prove that the function $\psi : G \to G$ defined via $\psi(z) = z^k$ is a surjective homomorphism but not an isomorphism, where k is a fixed positive integer greater than 1.

*You do not need to prove that this is a group. It turns out that G is the torsion subgroup of $\mathbb{C} \setminus \{0\}$.