Homework 2

Abstract Algebra I

Complete the following problems.

Problem 1. Determine whether each of the following binary operations is (i) associative and (ii) commutative.

- (a) The operation \star on \mathbb{R} defined via $a \star b = a + b + ab$.
- (b) The operation \circ on \mathbb{Q} defined via $a \circ b = \frac{a+b}{5}$.
- (c) The operation \odot on $\mathbb{Z} \times \mathbb{Z}$ defined via $(a, b) \odot (c, d) = (ad + bc, bd)$.
- (d) The operation \circledast on $\mathbb{Q} \setminus \{0\}$ defined via $a \circledast b = \frac{a}{b}$.
- (e) The operation \odot on $\mathbb{R}/I := \{x \in \mathbb{R} \mid 0 \le x < 1\}$ defined via $a \odot b = a + b \lfloor a + b \rfloor$ (i.e., $a \odot b$ is the fractional part of a + b).

Problem 2. Determine which of the following sets are groups under the given operation. Justify your answer.

- (a) $\mathbb{Z}/n\mathbb{Z}$ under multiplication mod *n*.
- (b) Set of rational numbers in lowest terms whose denominators are odd under addition. *Note:* Since we can write 0 = 0/1, 0 is included in this set.
- (c) Set of rational numbers in lowest terms whose denominators are even together with 0 under addition.
- (d) Set of rational numbers of absolute value less than 1 under addition.
- (e) \mathbb{R}/I under \odot as defined in Problem 1(e).

Problem 3. Let $G = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$. Prove one of the following.

- (a) The set *G* is a group under addition.
- (b) If $H = G \setminus \{0\}$, then *H* is a group under multiplication.

Problem 4. Assume *G* is a group and let $x \in G$. Prove one of the following.

- (a) If $a, b \in \mathbb{Z}$, then $x^{a+b} = x^a x^b$.
- (b) If $a, b \in \mathbb{Z}$, then $(x^a)^b = x^{ab}$.

Don't forget to handle the case when either *a* or *b* is nonpositive.

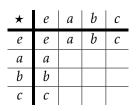
Problem 5. Assume *G* is a group and let $a, b \in G$. Is it true that $(ab)^n = a^n b^n$? If not, under what minimal conditions would it be true? Prove the statement that you think is true.

Problem 6. Assume *G* is a group. Prove that if $x^2 = e$ for all $x \in G$, then *G* is abelian.

Problem 7. Assume (G, \star) is a group and let H be a nonempty subset of G that is (i) closed under \star and (ii) closed under inverses (i.e., for all $h, k \in H$, (i) $hk \in H$ and (ii) $h^{-1} \in H$). Prove that H is a group under \star in its own right. Such a subset is called a *subgroup*.

Problem 8. Assume *G* is a group. Prove that if $x \in G$ such that $x^n \neq e$ for all $n \in \mathbb{Z}^+$, then $x^i \neq x^j$ for all $i \neq j$.

Problem 9. Assume $G = \{e, a, b, c\}$ is a group under \star with the property that $x^2 = x^4$ for all $x \in G$ (where *e* is the identity). Complete the following *group table*, where $x \star y$ is defined to be the entry in the row labeled by *x* and the column labeled by *y*.



Is your table unique? That is, did you have to fill it out the way you did? Deduce that *G* is abelian.

Problem 10. (Optional) Assume *G* is a finite group. Prove that every element of *G* must appear exactly once in every row and column of the group table for *G*. (Of course, we are not counting the row and column headings.)