Homework 4

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Problem 1. Let G be the group determined by the table in Problem 9 on Homework 2 and consider the group $\mathbb{Z}_2 \times \mathbb{Z}_2$ (where the operation is given by addition mod 2 in each component). Prove that $G \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ by exhibiting a "color-matching" of the group tables for both groups.

Problem 2. Consider a nickel and a quarter that are side by side on a table (it doesn't really matter, but let's assume that the nickel starts on the left and the quarter is on the right and both are heads up). Let f denote the action of flipping the left coin over and let w denote the action of swapping the positions of the two coins. Let G denote the group of actions on the two coins that is generated by $\{f, w\}$. That is, $G = \langle f, w \rangle$.

(a) Draw the Cayley diagram for G using the generating set $\{f, w\}$. *Hint:* There are 8 possible rearrangements of the coins and we can obtain all 8 using combinations of f and w.

(b) Using your Cayley diagram as justification, to what group is G isomorphic?

Problem 3. Let $\phi : G_1 \to G_2$ be a homomorphism.

(a) Prove that $\phi(e_1) = e_2$, where e_i is the identity of G_i.

(b) Prove that $\phi(x^n) = \phi(x)^n$ for all $x \in G_1$ and all $n \in \mathbb{Z}^+$.

(c) Prove that $\phi(x^{-1}) = \phi(x)^{-1}$ for all $x \in G_1$.

(d) Prove that $\phi(x^n) = \phi(x)^n$ for all $x \in G$ and all $n \in \mathbb{Z}$.

Problem 4. Let $\phi : G_1 \to G_2$ be an isomorphism.

(a) Prove that $|\phi(x)| = |x|$ for all $x \in G_1$.

(b) Prove that G_1 and G_2 have the same number of elements of order n for each $n \in \mathbb{Z}^+$.

(c) Is part (b) true if ϕ is only assumed to be a homomorphism? Justify your answer.

Problem 5. Complete any 3 of the following.

(a) Prove that $\mathbb{Z}_2 \times \mathbb{Z}_2$ is not isomorphic to $\mathbb{Z}_4$.

(b) Prove that D_8 and Q_8 are not isomorphic.

(c) Prove that the multiplicative groups $\mathbb{R} \setminus \{0\}$ and $\mathbb{C} \setminus \{0\}$ are not isomorphic.

(d) Prove that the additive groups $\mathbb{R}$ and $\mathbb{Q}$ are not isomorphic.

(e) Prove that the additive groups $\mathbb{Z}$ and $\mathbb{Q}$ are not isomorphic.

Problem 6. For each of the following pairs of groups, determine whether the given bijective function is an isomorphism from the first group to the second group. Justify your answers. (You may assume that each function is a bijection.)

(a) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$, $\phi(n) = n + 1$.

(b) $(\mathbb{Z}, +)$ and $(\mathbb{Z}, +)$, $\phi(n) = -n$.

(c) $(\mathbb{Q}, +)$ and $(\mathbb{Q}, +)$, $\phi(x) = x/2$.

Problem 7. Prove that the groups $(\mathbb{R}, +)$ and $(\mathbb{R}^+, \cdot)$ are isomorphic.

Problem 8. Let $\phi : G_1 \to G_2$ be a homomorphism.

(a) Prove if ϕ is an isomorphism, then G_1 is abelian iff G_2 is abelian.

(b) What conditions on ϕ are sufficient to ensure that if G_1 is abelian, then G_2 is abelian, as well. Justify your answer.

Problem 9. Let $\phi : G_1 \to G_2$ be a homomorphism. Prove that the image $\phi(G_1)$ is a subgroup of G_2. Quickly deduce that if ϕ is an injection, then $G_1 \cong \phi(G_1)$.

Problem 10. Let $\phi : G_1 \to G_2$ be a homomorphism. Define the *kernel* of ϕ to be the set

$$\ker(\phi) = \{g \in G_1 \mid \phi(g) = e_2\},$$

where e_2 is the identity in G_2.

(a) Why is $\ker(\phi)$ always nonempty?

(b) Prove that $\ker(\phi)$ is a subgroup of G_1.

(c) Prove that ϕ is an injection iff $\ker(\phi) = \{e_1\}$, where e_1 is the identity in G_1.

Problem 11. Let G be a group and let $\mathrm{Aut}(G)$ be the set of all isomorphisms from G onto G. Prove that $\mathrm{Aut}(G)$ is a group under function composition. Note that $\mathrm{Aut}(G)$ is called the *automorphism group of G* and its elements are referred to as *automorphisms*. There are some details to be shown for this problem, so don't cut too many corners.