Homework 6

Abstract Algebra I

Complete the following problems. Note that you should only use results that we've discussed so far this semester.

Problem 1. Determine whether each of the specified subsets is a subgroup of the given group. If the subset is a subgroup, prove it. If the subset is not a subgroup, explain why.

(a) The set of reflections from D_{2n}.

(b) $\{a + ai \mid a \in \mathbb{R}\} \subseteq \mathbb{C}$ (under addition).

(c) $\{z \in \mathbb{C} \mid |z| = 1\} \subseteq \mathbb{C} \setminus \{0\}$ (under multiplication).

(d) $\{x \in \mathbb{R} \mid x^2 \in \mathbb{Q}\} \subseteq \mathbb{R}$ (under addition).

Problem 2. Suppose H and K are subgroups of G. Prove that $H \cap K$ is also a subgroup of G.

Problem 3. Given an example of an infinite group G and an infinite subset H of G such that H is closed under the operation of G but is not a subgroup of G.

Problem 4. Let G be an abelian group.

(a) Prove that $\{g \in G \mid |g| < \infty\}$ is a subgroup of G (called the *torsion subgroup of* G).

(b) Give an example of a group G where the set described above is not a subgroup. Briefly justify your answer. *Hint:* Fiddle around some infinite non-abelian groups.

Problem 5. For each of the following groups, compute the centralizers of each element and find the center of each group.

(a) S_3

(b) D_8

(c) Q_8

Problem 6. Compute the normalizer for each subgroup of D_8. *Note:* There are 10 subgroups of D_8.

Problem 7. Let $H \leq G$. Prove one of the following.

(a) Prove that $H \leq N_G(H)$.

(b) Prove that $H \leq C_G(H)$ iff H is abelian.

Problem 8. Determine whether each group is cyclic. Justify your answer.

(a) $\mathbb{Z} \times \mathbb{Z}$.

(b) $\mathbb{Q}$

Problem 9. Provide an example of a group G such that every proper subgroup of G is cyclic, but G is not cyclic.

Problem 10. Give an example of two sets A and B contained in a group G such that (i) $A \subseteq B$, (ii) $A \neq B$, and (iii) $\langle A \rangle = \langle B \rangle$.

Problem 11. Let C_n be a cyclic group of order n. Fix $a \in \mathbb{Z}$. Define $\sigma_a : C_n \to C_n$ via $\sigma_a(x) = x^a$. Prove that σ_a is an automorphism of C_n iff a and n are relatively prime.

Problem 12. Prove one of the following.

(a) Prove that the subgroup $\langle (1,2), (1,3)(2,4) \rangle$ of S_4 is isomorphic to D_8.

(b) Prove that the subgroup $\langle s, r^2 \rangle$ of D_8 is isomorphic to $\mathbb{Z}_2 \times \mathbb{Z}_2$.

Problem 13. Consider the group $\mathbb{Z}/48\mathbb{Z}$.

(a) Find all generators for $\mathbb{Z}/48\mathbb{Z}$.

(b) What is the order of $\overline{30}$ in $\mathbb{Z}/48\mathbb{Z}$?

(c) Draw the subgroup lattice for $\mathbb{Z}/48\mathbb{Z}$.