$$S_{n+1}(t) = (1-t)^{n+2} \sum_{k\geq 0} (k+1)^{n+1} t^k$$

$$S_n(t) = (1-t)^{n+1} \sum_{k\geq 0} (k+1)^n t^k \qquad S_n'(t) = (1-t)^{n+1} \sum_{k\geq 0} k(k+1)^n t^{k-1} - (n+1)(1-t)^n \sum_{k\geq 0} (k+1)^n t^k$$

$$S_{n+1}(t) = (1+nt) S_n(t) + t(1-t) S_n'(t).$$

$$\Downarrow$$

$$= (1+nt)(1-t)^{n+1} \sum_{k\geq 0} (k+1)^n t^k + t(1-t)\left[(1-t)^{n+1} \sum_{k\geq 0} k(k+1)^n t^{k-1} - (n+1)(1-t)^n \sum_{k\geq 0} (k+1)^n t^k\right]$$

$$= \sum_{k\geq 0} (1+nt)(1-t)^{n+1}(k+1)^n t^k + \left[t(1-t)^{n+2} \sum_{k\geq 0} k(k+1)^n t^{k-1} - t(n+1)(1-t)^{n+1} \sum_{k\geq 0} (k+1)^n t^k\right]$$

$$= \sum_{k\geq 0} (1+nt)(1-t)^{n+1}(k+1)^n t^k + \sum_{k\geq 0} t(1-t)^{n+2} k(k+1)^n t^{k-1} - \sum_{k\geq 0} t(n+1)(1-t)^{n+1}(k+1)^n t^k$$

$$= \sum_{k\geq 0} \left[(1+nt)\underline{(1-t)^{n+1}}\, \underline{\underline{(k+1)^n}}\, t^k + \underline{(1-t)^{n+2}}\, k\underline{\underline{(k+1)^n}}\, t^k - (n+1)\underline{(1-t)^{n+1}}\, \underline{\underline{(k+1)^n}}\, t^{k+1} \right]$$

$$= \sum_{k\geq 0} (1-t)^{n+1}(k+1)^n t^k \left[(1+nt) + k(1-t) - (n+1)t\right]$$

$$= \sum_{k\geq 0} (1-t)^{n+1}(k+1)^n t^k \left[1 + \cancel{nt} + k - kt - \cancel{nt} - t\right]$$

$$= \sum_{k\geq 0} (1-t)^{n+1}(k+1)^n t^k \left[1(1-t) + k(1-t)\right]$$

$$= \sum_{k\geq 0} (1-t)^{n+1}(k+1)^n t^k (k+1)(1-t)$$

$$= \sum_{k\geq 0} (1-t)^{n+2}(k+1)^{n+1} t^k$$

$$= (1-t)^{n+2} \sum_{k\geq 0} (k+1)^{n+1} t^k$$

$$= S_{n+1}(t)$$