Homework 5

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Prove that $\mathbb{Z}[2i]$ is not a UFD.

Problem 2. Let $p \in \mathbb{Z}$ be prime and let $f(x) \in \mathbb{Z}[x]$. Determine general conditions under which $(p, f(x))_{\mathbb{Z}[x]}/(p)_{\mathbb{Z}[x]}$ is isomorphic to $(f(x))_{\mathbb{Z}/p\mathbb{Z}[x]}$ and prove that your answer is correct.

Problem 3. Identify the the following rings. That is, describe them in simpler terms.

(a) $\mathbb{Z}[x]/(2, 2x-1)$

(b) $\mathbb{Z}[x]/(4, 2x-1)$

Problem 4. Consider $p(x) = x^3 + 9x + 6 \in \mathbb{Q}[x]$.

(a) Show that $p(x)$ is irreducible in $\mathbb{Q}[x]$.

(b) If θ is a root of $p(x)$, find the inverse of $1+\theta$ in $\mathbb{Q}(\theta)$.

Problem 5. Consider $p(x) = x^3 + x + 1 \in \mathbb{Z}/2\mathbb{Z}[x]$.

(a) Show that $p(x)$ is irreducible in $\mathbb{Z}/2\mathbb{Z}[x]$.

(b) If θ is a root of $p(x)$, computer the powers of θ in $\mathbb{Z}/2\mathbb{Z}(\theta)$.

Problem 6. Prove that $x^5 - ax - 1 \in \mathbb{Z}[x]$ is irreducible unless $a = 0, 2$, or -1. The first two correspond to linear factors, the third corresponds to the factorization $(x^2 - x + 1)(x^3 + x^2 - 1)$.