Homework 7

Abstract Algebra II

Complete the following problems. Note that you should only use results that we've discussed so far this semester or last semester.

Problem 1. Determine the degree of the extension $\mathbb{Q}(2+\sqrt{3})/\mathbb{Q}$.

Problem 2. Let F be a field such that $\mathrm{Char}(F) \neq 2$. Let $d_1, d_2 \in F$ such that neither is a square in F. Prove that $F(\sqrt{d_1}, \sqrt{d_2})$ is an extension of degree 4 over F if $d_1 d_2$ is not a square in F and is of degree 2 otherwise.

Problem 3. Let K_1 and K_2 be finite extensions of F contained in the field K, and assume that both are splitting fields over F. Prove that $K_1 \cap K_2$ is a splitting field over F.

Problem 4. Prove that if $[F(\alpha) : F]$ is odd, then $F(\alpha) = F(\alpha^2)$.

Problem 5. Let p be a prime and consider the polynomial $f(x) = x^p - x + a \in \mathbb{Z}/p\mathbb{Z}[x]$, where $a \neq 0$.

(a) Prove that $f(x)$ is separable over $\mathbb{Z}/p\mathbb{Z}$.

(b) Prove that if α is a root of $f(x)$, then $\alpha + 1$ is also a root. Deduce that $f(x)$ is irreducible.

Problem 6. Suppose K is a field of characteristic p such that K contains an element that is not a pth power in K.

(a) Prove that there exist irreducible inseparable polynomials over K.

(b) Prove that there exist inseparable finite extensions of K.